1 Two Coupled Oscillators

1.1

\[ \hat{H}\psi_{00} = \left[ A + \epsilon \left( \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \right) - J \left( \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1 \right) \right] \psi_{00} \]

\[ = A\psi_{00} + \epsilon \hat{a}_1^\dagger \hat{a}_1 \psi_{00} + \epsilon \hat{a}_2^\dagger \hat{a}_2 \psi_{00} - J\hat{a}_1^\dagger \hat{a}_2 \psi_{00} - J\hat{a}_2^\dagger \hat{a}_1 \psi_{00} \]

\[ = A\psi_{00} \]  

This means that \( \psi_{00} \) is an eigenstate of \( \hat{H} \), with eigenvalue \( A \).

1.2

\[ \psi_{10} = a_1^\dagger \psi_{00}. \]

We will want to see if \( \hat{H}\psi_{10} = \lambda \psi_{10} \) for some \( \lambda \). Before jumping in, note that:

\[ \hat{a}_1\psi_{10} = a_1a_1^\dagger \psi_{00} = \left( 1 + a_1^\dagger a_1 \right) \psi_{00} = \psi_{00} \]

\[ \hat{a}_2\psi_{10} = a_2a_1^\dagger \psi_{00} = a_1^\dagger a_2 \psi_{00} = 0. \]

Now:

\[ \hat{H}\psi_{10} = A\psi_{10} + \epsilon \hat{a}_1^\dagger \hat{a}_1 \psi_{10} + \epsilon \hat{a}_2^\dagger \hat{a}_2 \psi_{10} - J\hat{a}_1^\dagger \hat{a}_2 \psi_{10} - J\hat{a}_2^\dagger \hat{a}_1 \psi_{10} \]

\[ = (A + \epsilon) \psi_{10} - J\hat{a}_2^\dagger \psi_{00} \]

\[ = (A + \epsilon) \psi_{10} - J\psi_{01}. \]

\( \psi_{10} \) is therefore not an eigenstate of the Hamiltonian: a particle initially in state \( \psi_{10} \) will evolve in to a linear combination of \( \psi_{01} \) and \( \psi_{10} \).

1.3

\[ i\partial_t \psi = \hat{H}\psi \]

\[ = a(t)\hat{H}\psi_{10} + b(t)\hat{H}\psi_{01}. \]
We found $\hat{H}\psi_{10}$ already. By symmetry, we can see that $\hat{H}\psi_{01}$ is the same as the expression for $\psi_{10}$ but with the labels 1 and 0 swapped:

$$\hat{H}\psi_{01} = (A + \epsilon)\psi_{01} - J\psi_{10}. \quad (12)$$

Therefore:

$$i\partial_t\psi = \hat{H}\psi$$

$$= [a(t) (A + \epsilon) - Jb(t)] \psi_{10} + [b(t) (A + \epsilon) - Ja(t)] \psi_{01}. \quad (14)$$

By isolating coefficients for $\psi_{10}$ and $\psi_{01}$ (which are orthonormal), we can extract the two first-order differential equations:

$$i\hbar\partial_t a(t) = (A + \epsilon) a(t) - Jb(t) \quad (15)$$

$$i\hbar\partial_t b(t) = (A + \epsilon) b(t) - Ja(t). \quad (16)$$

The simplest way to solve this, as with the ammonia molecule, is to find the eigenvectors and eigenvalues of the Hamiltonian matrix.

### 1.4

Let me write the Hamiltonian in four terms:

$$\hat{H} = A + \epsilon\hat{N} - J \left( \hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1 \right). \quad (17)$$

Clearly $\hat{N}$ commutes with the first two terms, so let us focus on the last two:

$$\left[ \hat{N}, \hat{a}_1^\dagger\hat{a}_2 \right] = \left[ \hat{N}_1, a_1^\dagger a_2 \right] + \left[ \hat{N}_2, a_1^\dagger a_2 \right]$$

$$= a_1^\dagger a_2 - a_1^\dagger a_2$$

$$= 0. \quad (20)$$

This is because the operator $\hat{a}_1^\dagger\hat{a}_2$ increases particle number 1 by one, and decreases particle number 2 by one, so that the sum of the two particle numbers is conserved. Similarly for the last term. Therefore, $\left[ \hat{N}, \hat{H} \right] = 0$.

### 2 Magnons

#### 2.1

$$\mathbf{S}_i \cdot \mathbf{S}_{i+1} = S^x_i S^x_{i+1} + S^y_i S^y_{i+1} + S^z_i S^z_{i+1}. \quad (21)$$

Now, recall that:

$$S^x = (S^+ + S^-)/2 \quad (22)$$

$$S^y = -i (S^+ - S^-)/2. \quad (23)$$

2
If your forget these as I did, one way to remember is to recall the matrix forms:

\[
S^x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S^y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\] (24)

It follows that:

\[
S_i \cdot S_{i+1} = \frac{1}{4} (S^+_i S^+_{i+1} + S^-_i S^-_{i+1} + S^+_i S^-_{i+1} + S^-_i S^+_{i+1})
- \frac{1}{4} (S^+_i S^-_{i+1} + S^-_i S^+_{i+1} - S^+_i S^-_{i+1} - S^-_i S^+_{i+1}) + S^z_i S^z_{i+1}
= \frac{1}{2} (S^+_i S^+_{i+1} + S^-_i S^-_{i+1}) + S^z_i S^z_{i+1}.
\] (25)

2.2

The terms in the operator \((S^+_i S^-_{i+1} + S^-_i S^+_{i+1})/2\) raises the spin on one site and lowers the spin on a neighbouring site. It will therefore destroy a state when it acts on two neighbouring sites with the same spin. In other words

\[
\frac{1}{2} \sum_i (S^+_i S^-_{i+1} + S^-_i S^+_{i+1}) |j\rangle = \frac{1}{2} \left( S^+_{j-1} S^-_j + S^-_j S^+_{j+1} \right) |j\rangle = \frac{|j+1\rangle + |j-1\rangle}{2},
\] (28)

and so this part of the Hamiltonian lowers the spin on site \(j\), while raising it on the neighbouring sites. The part of the Hamiltonian given by \(\sum_i S^z_i S^z_{i+1}\) will change the state only by a multiplicative constant, since all of the sites are in eigenstates of the \(S^z\) operator. To figure out what this multiplicative constant will be, note that \(S^z_i S^z_{i+1} = 1/4\) if both spins \(i\) and \(i+1\) are in the same direction, and \(S^z_i S^z_{i+1} = -1/4\) if the two spins are in opposite directions. Therefore, \(\sum_i S^z_i S^z_{i+1} |j\rangle = (n-3)/4 - 2/4,\) and:

\[
H |j\rangle = -J \sum_{i=1}^{n-1} S_i \cdot S_{i+1} |j\rangle = -\frac{J}{2} (|j+1\rangle + |j-1\rangle) - J \frac{n-5}{4} |j\rangle
\] (29)

2.3

Writing the state vector for a generic ‘one-particle’ state \(|\psi\rangle = \sum_i c_i |i\rangle\), or as a column vector:

\[
|\psi\rangle = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},
\] (30)
the Hamiltonian becomes:

\[
H = \frac{-J}{4} \begin{pmatrix}
  n-5 & 2 & 0 & 0 \\
  2 & n-5 & 2 & 0 \\
  0 & 2 & n-5 & 2 \\
  0 & 0 & 2 & n-5 \\
\vdots & & & \ddots
\end{pmatrix}.
\] (31)

An upward-pointing spin will travel backwards and forwards along the spin-chain like a free particle moving in one dimension. ‘Really’, the particles are the stationary sites, and their spins are behaving collectively. But this collective motion of spins has all the properties of a particle: it has a mass, a momentum, quantized energy. It is most usefully thought of as a particle.

3 Parameters in our model

The Hamiltonian can be written as:

\[
H = K + V_\kappa + V_\gamma
\]

(32)

3.1 \(K\)

\[
K = \sum_j \frac{p_j^2}{2m}
\]

(33)

\[
= -\frac{\hbar^2}{4d^2m} \sum_j \left[ (a_j - a_j^\dagger) - \alpha (a_{j+1} - a_{j+1}^\dagger + a_{j-1} - a_{j-1}^\dagger) \right]^2
\]

(34)

\[
= -\frac{\hbar^2}{4d^2m} \sum_j \left[ (a_j - a_j^\dagger)^2 - \alpha (a_j - a_j^\dagger) (a_{j+1} - a_{j+1}^\dagger + a_{j-1} - a_{j-1}^\dagger) \right.
\]

\[
- \alpha (a_{j+1} - a_{j+1}^\dagger + a_{j-1} - a_{j-1}^\dagger) (a_j - a_j^\dagger) + \mathcal{O} (\alpha^2)
\]

\[
= -\frac{\hbar^2}{4d^2m} \sum_j \left[ (a_ja_j + a_j^\dagger a_j^\dagger) - (a_ja_j + a_j^\dagger a_j) - 2\alpha (a_j - a_j^\dagger) (a_{j+1} - a_{j+1}^\dagger + a_{j-1} - a_{j-1}^\dagger) \right].
\]

(36)
The term proportional to $\alpha$ can be written:

$$\alpha \text{ term } = \frac{\hbar^2}{8d^2m} \sum_j 2\alpha \left( a_j - a_j^\dagger \right) \left( a_{j+1} - a_{j+1}^\dagger + a_{j-1} - a_{j-1}^\dagger \right)$$

\begin{align*}
\text{(37)} &= \frac{\alpha \hbar^2}{2d^2m} \left[ \sum_j \left( a_j - a_j^\dagger \right) \left( a_{j+1} - a_{j+1}^\dagger \right) + \sum_j \left( a_j - a_j^\dagger \right) \left( a_{j-1} - a_{j-1}^\dagger \right) \right] \\
\text{(38)} &= \frac{\alpha \hbar^2}{2d^2m} \left[ \sum_j \left( a_j - a_j^\dagger \right) \left( a_{j+1} - a_{j+1}^\dagger \right) + \sum_j \left( a_{j+1} - a_{j+1}^\dagger \right) \left( a_j - a_j^\dagger \right) \right] \\
\text{(39)} &= \frac{\alpha \hbar^2}{2d^2m} \sum_j \left( a_j - a_j^\dagger \right) \left( a_{j+1} - a_{j+1}^\dagger \right) \\
\text{(40)} &= \sum_j \frac{\alpha \hbar^2}{d^2m} \left[ \left( a_j a_{j+1} + a_j^\dagger a_{j+1}^\dagger \right) - \left( a_j a_{j+1} + a_j^\dagger a_{j+1}^\dagger \right) \right]
\end{align*}

We have made liberal use of the relation $[a_{j+1}, a_j] = 0$. Therefore, we have:

$$A_K = \frac{\hbar^2}{4md^2}, \quad B_K = -\frac{\hbar^2 \alpha}{d^2m}, \quad C_K = -\frac{\hbar^2}{4d^2m}, \quad D_K = \frac{\alpha \hbar^2}{d^2m}. \quad (42)$$

### 3.2 $V_\kappa$

\begin{align*}
V_\kappa &= \sum_j \frac{\kappa x_j^2}{2} \\
\text{(43)} &= \frac{\kappa d^2}{4} \sum_j \left[ \left( a_j + a_j^\dagger \right) - \alpha \left( a_{j+1} + a_{j+1}^\dagger + a_{j-1} + a_{j-1}^\dagger \right) \right]^2 \\
\text{(44)} &= \frac{\kappa d^2}{4} \sum_j \left[ \left( a_j + a_j^\dagger \right)^2 + 2\alpha \left( a_j + a_j^\dagger \right) \left( a_{j+1} + a_{j+1}^\dagger + a_{j-1} + a_{j-1}^\dagger \right) + \mathcal{O}(\alpha^2) \right] \\
\text{(45)} &= \frac{\kappa d^2}{4} \sum_j \left[ \left( a_j a_{j+1} + a_j^\dagger a_{j+1}^\dagger \right) + \left( a_j a_j^\dagger + a_j^\dagger a_j \right) + 4\alpha \left( a_j a_{j+1} + a_j^\dagger a_{j+1}^\dagger \right) + 4\alpha \left( a_j a_{j+1} + a_j^\dagger a_{j+1} \right) \right].
\end{align*}

Therefore, we have:

$$K_{V_\kappa} = \frac{\kappa d^2}{4}, \quad B_{V_\kappa} = \kappa d^2 \alpha, \quad C_{V_\kappa} = \frac{\kappa d^2}{4}, \quad D_{V_\kappa} = \kappa d^2 \alpha. \quad (47)$$
3.3 \( V_γ \)

\[
V_γ = \sum_j \frac{γ}{2} (x_{j+1} - x_j)^2
\]  

\[
= \frac{κd^2}{4} \sum_j \left( a_{j+1} + a_{j+1}^\dagger - a_j - a_j^\dagger + O(α^2) \right)^2
\]  

\[
= \frac{κd^2}{2} \sum_j \left[ \left( a_j a_j^\dagger + a_j^\dagger a_j \right) - \left( a_{j+1} a_j + a_j a_{j+1} \right) + \left( a_j a_j^\dagger + a_j^\dagger a_j \right) \right] (50)
\]

Therefore, we have:

\[
K_{V_γ} = \frac{γd^2}{2}, \quad B_{V_γ} = -\frac{γd^2}{2}, \quad C_{V_γ} = \frac{γd^2}{2}, \quad D_{V_γ} = -\frac{γd^2}{2}. \quad (51)
\]

4 Classical Sound Waves

4.1

The two equations are:

\[
-\text{i}mωu = v
\]

\[
-\text{i}ωv = -u \left( κ + γ \left( 2 - e^{ika} - e^{-ika} \right) \right) = -u \left( κ + 2γ \left( 1 - \cos (ka) \right) \right). \quad (53)
\]

Plugging \( u \) from the first into the second gives:

\[
-\text{i}ωv = \frac{v}{mωi} \left( κ + 2γ \left( 1 - \cos (ka) \right) \right) \quad (54)
\]

\[
ω^2 = \frac{1}{m} \left( κ + 2γ \left( 1 - \cos (ka) \right) \right) \quad (55)
\]

4.2

For small \( k \):

\[
ω^2 = \frac{1}{m} \left( κ + γk^2 a^2 \right) \quad (56)
\]

\[
E^2 = ℏω^2 = \frac{h^2κ}{m} + \frac{γa^2}{m} h^2 k^2. \quad (57)
\]

Now, we know that \( E^2 = p^2c^2 + M^2c^4 = h^2k^2c^2 + M^2c^4 \) (typo in question sheet), and so from the second term we see that:

\[
c^2 = \frac{γa^2}{m} \quad (58)
\]

and the first term gives us:

\[
M^2 = \frac{ℏ^2mk}{γ^2a^4}. \quad (59)
\]

This mass gap means that there is a minimum energy we need to put in to excite any optical modes, as opposed to say, photons which can be produced with arbitrarily small energy (and arbitrarily large wavelength).