Problem 1. Ion Detector

1.1
Yes, the device can simultaneously measure the x- and y-coordinates. This is equivalent to the mathematical statement $[x, y] = 0,$ i.e. if two operators commute then one can simultaneously measure the quantities associated with these operators.

1.2

(a) Let’s call $S_1$ the disk of radius $d$ centred at the origin and $S_2$ the disk of radius $d$ centred at $(a, 0)$. Notice that since $a > 2d$, $S_1$ and $S_2$ do not overlap. Since $\phi_1(x, y)$ is nonzero only in $S_1$ and $\phi_2(x, y)$ only in $S_2$, this implies that they do not overlap and so $\phi_1^*(x, y)\phi_2(x, y) = 0$.

The normalization condition then states

$$1 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\Psi(x, y)|^2 = \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy (\phi_1^* + \phi_2^*)(\phi_1 + \phi_2)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\phi_1|^2 + |\phi_2|^2$$

$$= \frac{1}{2} \int_{S_1} \frac{1}{\pi d^2} + \int_{S_2} \frac{1}{\pi d^2} = 1 \quad (1)$$

In the third step we have used the fact that $\phi_1$ and $\phi_2$ do not overlap to eliminate cross terms.

In the final step we have used the fact that the wavefunctions are constant in their respective disks, but zero outside.

(b) One count will be detected, since there is only one ion.

(c) The ion has a 50% chance of being detected on $S_1$ and a 50% chance of being detected on $S_2$.

To see this, the probability of being detected on $S_1$ is given by

$$P(S_1) = \int_{S_1} |\Psi|^2 = 0.5 \quad (2)$$

* Special thanks to Sergei Dyda
The same can be shown for $S_2$.

**Problem 2. Finite Differences in Time**

2.1

We are given

\begin{align*}
    i\psi_0'(t) &= -\psi_1(t) \\
    i\psi_1'(t) &= -\psi_0(t)
\end{align*}

(3) (4)

We can differentiate (13) one more time and substitute in (14), and vice versa, to obtain two uncoupled differential equations:

\begin{align*}
    \psi_0''(t) &= -\psi_0(t) \\
    \psi_1''(t) &= -\psi_1(t)
\end{align*}

(5) (6)

These are the simple harmonic oscillator equations, so we expect the most general solution to be of the form:

\begin{align*}
    \psi_0(t) &= Pe^{it} + Qe^{-it} \\
    \psi_1(t) &= Re^{it} + Se^{-it}
\end{align*}

(7) (8)

where $P$, $Q$, $R$ and $S$ are complex coefficients. (13) and (14) implies that $P = R$ and $Q = -S$.

The initial conditions then imply that

\begin{align*}
    P + Q &= 1 \\
    P - Q &= 0
\end{align*}

(9) (10)

So we find that $P = Q = \frac{1}{2}$. Therefore, the solutions are

\begin{align*}
    \psi_0(t) &= \frac{1}{2}e^{it} + \frac{1}{2}e^{-it} = \cos(t) \\
    \psi_1(t) &= \frac{1}{2}e^{it} - \frac{1}{2}e^{-it} = i\sin(t)
\end{align*}

(11) (12)

2.2

We rewrite (13) and (14) using finite difference as

\begin{align*}
    i \frac{\psi_0(t + \delta t) - \psi_0(t)}{\delta t} &= -\psi_1(t) \\
    i \frac{\psi_1(t + \delta t) - \psi_1(t)}{\delta t} &= -\psi_0(t)
\end{align*}

(13) (14)
A little bit of manipulation gives you
\[
\begin{pmatrix}
\psi_0(t + \delta t) \\
\psi_1(t + \delta t)
\end{pmatrix}
= \begin{pmatrix} 1 & i\delta t \\ i\delta t & 1 \end{pmatrix}
\begin{pmatrix}
\psi_0(t) \\
\psi_1(t)
\end{pmatrix}
\]  \hspace{1cm} (15)

Therefore,
\[
\mathbb{M} = \begin{pmatrix} 1 & i\delta t \\ i\delta t & 1 \end{pmatrix}
\]  \hspace{1cm} (16)

2.3

We want to find the right eigenvectors and eigenvalues of \( \mathbb{M} \). We first solve
\[
0 = |\mathbb{M} - x\mathbb{I}| = \begin{vmatrix}
1 - x & \delta t \\
\delta t & 1 - x
\end{vmatrix}
= (1 - x)^2 + (\delta t)^2
\]  \hspace{1cm} (17)

for \( x \) to find the eigenvalues. We find two solutions
\[
\lambda = 1 + i\delta t \\
\lambda' = 1 - i\delta t
\]  \hspace{1cm} (18)

The eigenvectors (note: not normalised) are
\[
\begin{pmatrix}
a_0 \\
a_1
\end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\text{ for } \lambda
\]
\[
\begin{pmatrix}
b_0 \\
b_1
\end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\text{ for } \lambda'
\]  \hspace{1cm} (19)

2.4

\[
\begin{pmatrix}
\psi_0(0) \\
\psi_1(0)
\end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
= A \begin{pmatrix} 1 \\ 1 \end{pmatrix} + B \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]  \hspace{1cm} (20)

We then find that \( A = B = \frac{1}{2} \).

2.5 Using the formula given in the question, we find that
\[
\begin{pmatrix}
\psi_0(N\delta t) \\
\psi_1(N\delta t)
\end{pmatrix} = \frac{1}{2} (1 + i\delta t)^N \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} (1 - i\delta t)^N \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]  = \frac{1}{2} \begin{pmatrix}
(1 + i\delta t)^N + (1 - i\delta t)^N \\
(1 + i\delta t)^N - (1 - i\delta t)^N
\end{pmatrix}
\]  \hspace{1cm} (21)
2.5

I am not an expert on Python, but here’s my (feeble) attempt to use it to generate the plots. Notice that for the forward Euler approximation, the magnitude of $|\psi_0|^2$ increases with time, although the increase is slower for smaller $\delta t$. The increase occurs due to the fact that the matrix $M$ is nonunitary.

![Plots showing magnitude of $|\psi_0|^2$ over time for different $\delta t$.]

Problem 3. Gaussian integrals

3.1

$$\int_{-\infty}^{\infty} dx \, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$  \hspace{1cm} (22)

See any standard mathematics textbook for the derivation.

3.2

Let $x' = x - \frac{a}{2\alpha}$.

$$\int_{-\infty}^{\infty} dx \, e^{-\alpha(x-x')^2} = \int_{-\infty}^{\infty} d\left(x' + \frac{a}{2\alpha}\right) e^{-\alpha x'^2}$$
$$= \int_{-\infty}^{\infty} dx' \, e^{-\alpha x'^2}$$
$$= \sqrt{\frac{\pi}{\alpha}}$$  \hspace{1cm} (23)

3.3

$$\int_{-\infty}^{\infty} dx \, e^{-\alpha x^2} - ax = \int_{-\infty}^{\infty} dx \, e^{-\alpha(x+\frac{a}{2\alpha})^2 + \frac{a^2}{4\alpha}}$$
$$= e^{\frac{a^2}{4\alpha}} \int_{-\infty}^{\infty} dx \, e^{-\alpha(x+\frac{a}{2\alpha})^2}$$
$$= e^{\frac{a^2}{4\alpha}} \sqrt{\frac{\pi}{\alpha}}$$  \hspace{1cm} (24)
In the first step, we completed the square for the exponent. In the last step, we used the results from Part 3.2.

3.4

\[ e^{\frac{\alpha^2}{4\alpha}} \sqrt{\frac{\pi}{\alpha}} = \sqrt{\frac{\pi}{\alpha}} \left( 1 + \frac{\alpha^2}{4\alpha} + \frac{1}{2!} \left( \frac{\alpha^2}{4\alpha} \right)^2 + \ldots + \frac{1}{n!} \left( \frac{\alpha^2}{4\alpha} \right)^n + \ldots \right) \] (25)

3.5

We are given

\[ \int_{-\infty}^{\infty} dx e^{-\alpha x^2 - ax} = \sum_j j! \int_{-\infty}^{\infty} dx x^j e^{-\alpha x^2} \] (26)

We now compare this with the results we got from Part 3.4.

\[ \sqrt{\frac{\pi}{\alpha}} \frac{1}{n!} (\alpha)^n = \frac{a^{2n}}{(2n)!} \int_{-\infty}^{\infty} dx x^{2n} e^{-\alpha x^2} \] (27)

This implies that

\[ I_{2n} = \int_{-\infty}^{\infty} dx x^{2n} e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}} \frac{1}{n!} \frac{1}{(4\alpha)^n} \] (28)

It is also easy to show that \( I_{2n+1} = 0 \).

Problem 4. Delta Function

4.1

\[ I_s = \int_{-\infty}^{\infty} dx f(x) \delta_s(x) = \int_{-s/2}^{s/2} (f_0 + f_1 x + f_2 x^2) \frac{1}{s} \] (29)

\[ = f_0 + \frac{f_2 s^2}{12} \]

4.2

\[ \lim_{s \to 0} I_s = \lim_{s \to 0} (f_0 + \frac{f_2 s^2}{12}) = f_0 \] (30)

4.3

\[ J_s = \int_{-\infty}^{\infty} dx \delta_s(x)^2 = \int_{-s/2}^{s/2} \frac{1}{s^2} = \frac{1}{s} \] (31)

4.4

\[ \lim_{s \to 0} \frac{1}{s} \] diverges (32)
4.5 Assume $a > 0$. We will use a change of variables:

$$\int f(x)\delta(ax)dx = \int f(y/a)\delta(y){1\over a}dy = {1\over a}f(0) = \int f(x){1\over a}\delta(x)dx$$

(33)

4.6

4.7

$$\tilde{\delta}_s(x = 0) = {1 \over \pi s}$$

(34)

4.8

Setting $1 / (2\pi s) = {s \over \pi(x^2+s^2)}$ yields $x = s = 0.01$.

Width of $\tilde{\delta}_s(x)$ is $2s$.

4.9

Use whatever method you want (look up integrals, mathematica, DIY):

$$\tilde{I}_s \simeq f(0) \int dx \tilde{\delta}_s(x) = f(0)$$

(35)

4.10
4.11

Again, evaluate the integral:

\[ \tilde{I}_s \simeq f(0) \int dx \tilde{\delta}_s(x) = f(0) \]  

(36)