Problem 1. Symmetric Bell’s Inequalities

1.1
All the possibilities and their probabilities have been listed, so their probabilities should add to 1.

\[ 2p + 6c = 1 \]  
\[ (1) \]

1.2
Since \( p \geq 0 \), we have

\[ c = \frac{1 - 2p}{6} \leq \frac{1}{6} \]  
\[ (2) \]

1.3
Let us choose the first detector to be in the \( \hat{z} \) direction and the second detector in the \( 120^\circ \) direction. The coincidence probability is then given by

\[ P_{\text{coinc}} = P(\uparrow, \uparrow, \uparrow) + P(\uparrow, \uparrow, \downarrow) + P(\downarrow, \downarrow, \uparrow) + P(\downarrow, \downarrow, \downarrow) \]
\[ = 2p + 2c \]
\[ = 1 - 4c \]
\[ \geq \frac{1}{3} \]  
\[ (3) \]

We see that the minimum coincidence probability is 33%, whereas quantum mechanics allows for a coincidence probability of 25% for a completely correlated state.

Problem 2. Entanglement

2.1
The product state given in the question can be written as

\[ |\psi\rangle = ac|\uparrow\uparrow\rangle + bc|\downarrow\uparrow\rangle + ad|\uparrow\downarrow\rangle + bd|\downarrow\downarrow\rangle \]  
\[ (4) \]

A Bell state requires \( ac = bd = \frac{1}{\sqrt{2}} \) and \( ad = bc = 0 \). It is clear that no solutions for \( a, b, c \) and \( d \) exist. For example, we find that \( ac \times bd = abcd = \frac{1}{2} \) but \( bc \times ad = abcd = 0 \), which is a contradiction.
2.2
Another example of an entangled state is

\[ |\psi\rangle = \frac{1}{\sqrt{2}} (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) \]  

(5)

This state requires \( ad = bc = \frac{1}{\sqrt{2}} \) and \( ac = bd = 0 \), which again leads to a contradiction.

Problem 3. Shor’s Algorithm

3.1
However you do it, you should find that the period is 80.

3.2
You should find that for \( r = 80 \), \( f(r/2) - 1 = 66 \) and \( f(r/2) + 1 = 68 \). 66 and 187 have common factors 11, while 68 and 187 have common factors 17. Since 187 = 11 \( \times \) 17, we have indeed factorised 187.

Problem 4.

4.1
From dimensional analysis, we can guess that it might be

\[ V_{\text{max}} = \frac{kQq}{a} \]  

(6)

where \( Q \) is the charge of the nucleus (+79e for gold) and \( q \) is the charge of the \( \alpha \) particle (+2e).

Let us compare to what we get from a more ‘rigourous’ derivation. Let’s assume the charge of the nucleus to be uniformly distributed. Using Gauss law, we find that

\[ V(r) = \begin{cases} \frac{kQq}{r} & \text{if } r > a \\ \frac{kQq}{2a} (3 - \frac{r^2}{a^2}) & \text{if } r \leq a \end{cases} \]  

(7)

Hence, the potential is maximum at the origin, with a value of \( \frac{3kQ}{2a} \), which certainly agrees with our guess within an order of magnitude!

4.2
Generating the plot using Python. \( V_0 \) is defined as \( kQq/a \).
4.3
Using classical reasoning, all the $\alpha$ particles would have passed through the nucleus without rebounding had the maximum potential energy been less than the kinetic energy. However, since we know that some $\alpha$ particles are repelled, this implies that $V_{\text{max}}$ is at least the kinetic energy, so 7.5 MeV is the lower bound on $V_{\text{max}}$. Hence we can calculate an upper bound on the 'radius' of the nucleus.

$$a \lesssim \frac{kQq}{V_{\text{max}}} \simeq 3 \times 10^{-14} \text{ m}$$ (8)

4.4
From the above equation, we see that he could have used more energetic particles to put a more stringent bound on $a$.

Problem 5.

5.1
$s = 2n\pi$ for all $n \in \mathbb{Z}$.

5.2
$s = 2n\pi + \theta$ for all $n \in \mathbb{Z}$.

5.3
Assuming that $A$ is positive: $s = 2n\pi + \phi$ for all $n \in \mathbb{Z}$, and $t = \ln(A)$. 
Problem 6.

The easiest way to think about this is just as a change in variables in multi-variable calculus. We originally have variables \((x, y)\) and we are changing to variables \((z, z^*)\) through the transformation \(z = x + iy\) and \(z^* = x - iy\). So think of \(x(z, z^*) = (z + z^*)/2\) and \(y(z, z^*) = -i(z - z^*)/2\). Then by chain rule we have

\[
\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z} = \frac{1}{2} \frac{\partial f}{\partial x} - \frac{i}{2} \frac{\partial f}{\partial y} \tag{9}
\]

Similarly we can show that

\[
\frac{\partial f}{\partial z^*} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z^*} = \frac{1}{2} \frac{\partial f}{\partial x} + \frac{i}{2} \frac{\partial f}{\partial y} \tag{10}
\]

By adding and subtracting these 2 equations we find

\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial z^*} \tag{11}
\]

\[
\frac{\partial f}{\partial y} = i \left( \frac{\partial f}{\partial z} - \frac{\partial f}{\partial z^*} \right) \tag{12}
\]

Hence, \(\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z^*} = 0\) implies \(\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0\).