Due Dec 9 2015, 4:30pm

You may use your notes or other books. You may not get help from other people.

Problem 1. Wavefunctions

1.1. Write a two-particle wavefunction which is bosonic.

1.2. Write a two-particle wavefunction which is fermionic.

1.3. Write a two-particle wavefunction which is neither bosonic nor fermionic.

1.4. Write a two-particle wavefunction in which the two particles are bound closely together, but their center of mass is delocalized.

Problem 2. Neutrino Oscillations

The 2015 Nobel Prize in Physics was awarded jointly to Takaaki Kajita and Arthur B. McDonald “for the discovery of neutrino oscillations, which show that neutrinos have mass.” This year’s Nobel lectures will be webcast on December 8 at http://www.kva.se/sv/Kalendarumlista/2015/the-nobel-lectures-2015/.

There are 3 flavors of neutrino: electron, muon and tau. The weak interaction produces neutrinos in these flavor eigenstates, $|e\rangle, |\mu\rangle, |\tau\rangle$, which can be represented as column vectors in flavor space:

$$|e\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$  \hspace{1cm} (1)

$$|\mu\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$  \hspace{1cm} (2)

$$|\tau\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$  \hspace{1cm} (3)

These are not eigenstates of the Hamiltonian. One model, consistent with the experimental data,
is that the energy eigenstates are:

\[
|1\rangle = \begin{pmatrix} 0.82 \\ -0.51 \\ 0.26 \end{pmatrix} \tag{4}
\]

\[
|2\rangle = \begin{pmatrix} 0.55 \\ 0.58 \\ -0.60 \end{pmatrix} \tag{5}
\]

\[
|3\rangle = \begin{pmatrix} 0.16 \\ 0.63 \\ 0.75 \end{pmatrix} \tag{6}
\]

with energies \( E_1 = \sqrt{(5\text{meV})^2 + p^2c^2} \), \( E_2 = \sqrt{(10\text{meV})^2 + p^2c^2} \), and \( E_3 = \sqrt{(50\text{meV})^2 + p^2c^2} \), where \( p \) is the neutrino momentum. We will assume \( p = 10\text{MeV}/c \), a typical solar neutrino momentum.

In the sun, electron neutrino’s are produced.

2.1. In terms of a length 3 column vector, what is the flavor wavefunction \( |\psi(t = 0)\rangle \) of a neutrino when it is first produced?

2.2. In terms of the energy eigenstates, this wavefunction can be written \( |\psi(t = 0)\rangle = \alpha|1\rangle + \beta|2\rangle + \gamma|3\rangle \). Find \( \alpha, \beta, \gamma \).

2.3. What is the flavor wavefunction at some later time \( t \), \( |\psi(t)\rangle \). Write your solution both as a sum of terms involving the energy eigenstates, and in the form of a length 3 column vector.

2.4. Assuming that the neutrinos move at the speed of light, how long do they take to travel from the sun to the earth? Call this time \( t \).

2.5. Calculate \( E_1t/\hbar \).

2.6. Calculate \( \phi_{21} = (E_2 - E_1)t/\hbar \). [Note: The easiest way to calculate this is to Taylor expand \( E_2 \) and \( E_1 \), taking \( pc \) large.]

2.7. Calculate \( \phi_{31} = (E_3 - E_1)t/\hbar \). [Note: The easiest way to calculate this is to Taylor expand \( E_3 \) and \( E_1 \), taking \( pc \) large.]

2.8. The flavor wavefunction when the neutrinos hit the earth can be represented as a length 3 column vector of the form

\[
|\psi(t)\rangle = e^{-iE_1t} \begin{pmatrix} a \\ b \\ c \end{pmatrix},
\]

where \( a, b \) and \( c \) are complex numbers. How is the probability that a neutrino will be in an electron flavor eigenstate related to \( a, b, c \)? Do not bother numerically calculating \( a, b, \) or \( c \). Due to the large numbers, and round-off errors, you can’t expect it to be accurate enough to be worth the effort.
2.9. Suppose we have a reactor which produces electron neutrinos with momentum $p = 10\text{MeV}/c$. They travel a distance of $d = 30\text{km}$ before hitting a detector. Calculate the travel time $t$, the phases $\phi_{21}$, $\phi_{31}$, and the amplitude $a$. Use these results to calculate the probability that the neutrinos are in an electron flavor eigenstate when they hit the detector.

Problem 3. Wave-nature of neutrons

Over the past few decades there have been a number of experiments which have observed the wave-nature of neutrons. [See for example, Nesvizhevsky et al. Nature 415, 297 (2002), available at http://www.nature.com/nature/journal/v415/n6869/abs/415297a.html.]

In a typical setup the neutrons are trapped by the gravitational potential and a reflecting surface. A reasonable model for the wavefunction of the neutron is a one-dimensional Schrödinger equation:

$$i\hbar \partial_t \psi(z, t) = -\frac{\hbar^2}{2m} \partial^2_z \psi(z, t) + mgz \psi(z, t), \quad (8)$$

with boundary condition $\psi(0, t) = 0$. Here $m$ is the neutron’s mass, and $g$ is the acceleration due to gravity. One can adimensionalize this equation, so that it reads

$$i\partial_{\tau} \psi(s, \tau) = -\frac{1}{2} \partial^2_s \psi(s, \tau) + s \psi(s, \tau), \quad (9)$$

where $s = z/z_0$ and $\tau = t/t_0$.

3.1. What are $z_0$ and $t_0$ in physical units ($\mu\text{m}$ and $s$)? What is the characteristic energy scale of the system $\epsilon_0 = \hbar/t_0$ in peV?

3.2. Consider the variational wavefunction

$$\psi(s) = A s e^{-\lambda s}, \quad (10)$$

where $A$ and $\lambda$ are parameters. Use the normalization condition on the wavefunction to determine $A$.

3.3. Use the variational principle and this wavefunction to estimate the ground state energy in units of $\epsilon_0$.

3.4. Write a computer program to calculate the ground state energy in units of $\epsilon_0$. Give your numerical answer here. Attach your program to the end of this exam. If you hand it in electronically, please be sure that your name is on it.

Problem 4. How big is a neutron star? We all know neutron stars are dense burned out stars. They are made from neutrons. It turns out that the only thing which is supporting them against gravitational collapse is the fact that neutrons are fermions.
4.1. Suppose the Neutron star contains \( N \) electrons, and has radius \( R \). The kinetic energy of the neutrons should be roughly the same as the kinetic energy of \( N \) neutrons in a 3D infinite square well of size \( R \times R \times R \). Use this argument to estimate the kinetic energy as a function of the radius of the star. Ignore relativistic effects, but include that fact that the neutrons are spin-1/2 fermions.

4.2. Estimate the gravitational potential energy of \( N \) neutrons forming a ball of radius \( R \). Make any approximations that you feel appropriate – just be explicit in your approximation.

4.3. Sketch the total energy of the neutron star as a function of radius. Label your axes.

4.4. Minimize the energy to derive a relationship between the radius of a neutron star and its mass.

4.5. A typical neutron star has a mass of about two solar masses. Using your size estimate, how large will this neutron star be? How does that compare to the size of the sun?

4.6. A typical nucleus has size \( r = 10^{-15} \text{m} \) – a lengthscale known as a “Fermi.” How does the density of a neutron star compare with that of a nucleus?

**Problem 5. Hyperfine Structure** You know a lot about the electronic excitations of hydrogen (and other atoms). One thing which is probably less familiar to you is the fact that the electronic spin can couple to the motion of the electron (so-called “spin-orbit coupling”). Such spin-orbit coupling is responsible for the small splitting between the lines in sodium. This splitting is so small that it is referred to as “fine structure”.

Here we are going to explore an even more subtle effect – namely that there is a coupling between the electronic and nuclear spin in hydrogen – leading to very-very-very small splittings of spectral lines – so-called hyper-fine structure. This splitting is in the microwave frequency band (\( \nu = 1.4 \text{ GHz}, \text{or } \lambda = 21\text{cm} \)). It is very important for astrophysics, as you can use it to detect clouds of hydrogen.

The electron and the proton in the hydrogen atom can each be in one of two states: giving a total of four possibilities:

1. \(|\uparrow\uparrow\rangle\): electron up, proton up
2. \(|\uparrow\downarrow\rangle\): electron up, proton down
3. \(|\downarrow\uparrow\rangle\): electron down, proton up
4. \(|\downarrow\downarrow\rangle\): electron down, proton down

We will let \( S_e \) and \( S_p \) be the spin operators for the electron and the proton.

**5.1.** What is \( (S_e)_z |\uparrow\rangle \)?
5.2. What is \((S_e)_x \uparrow \downarrow\)?

5.3. What is \((S_e)_z(S_p)_z \uparrow \downarrow\)?

5.4. By symmetry, the coupling between the electron spin and and the proton spin takes the form

\[
H = A \left[(S_e)_z(S_p)_z + (S_e)_y(S_p)_y + (S_e)_x(S_p)_x \right].
\] (11)

Acting on one of the wavefunctions,

\[
H|\uparrow\uparrow\rangle = a|\uparrow\uparrow\rangle + b|\uparrow\downarrow\rangle + c|\downarrow\uparrow\rangle + d|\downarrow\downarrow\rangle.
\] (12)

Find \(a, b, c, d\).

5.5. \[H|\uparrow\downarrow\rangle = e|\uparrow\uparrow\rangle + f|\uparrow\downarrow\rangle + g|\downarrow\uparrow\rangle + h|\downarrow\downarrow\rangle.\] (13)

Find \(e, f, g, h\).

5.6. \[H|\downarrow\uparrow\rangle = i|\uparrow\uparrow\rangle + j|\uparrow\downarrow\rangle + k|\downarrow\uparrow\rangle + l|\downarrow\downarrow\rangle.\] (14)

Find \(i, j, k, l\).

5.7. \[H|\downarrow\downarrow\rangle = m|\uparrow\uparrow\rangle + n|\uparrow\downarrow\rangle + o|\downarrow\uparrow\rangle + p|\downarrow\downarrow\rangle.\] (15)

Find \(m, n, o, p\).

5.8. Suppose \(|\psi\rangle = \alpha|\uparrow\uparrow\rangle + \beta|\uparrow\downarrow\rangle + \gamma|\downarrow\uparrow\rangle + \delta|\downarrow\downarrow\rangle\) is an energy eigenstate, with energy \(E\). Show that the coefficients \(\alpha, \beta, \gamma, \delta\) obey an equation

\[
\begin{pmatrix}
? & ? & ? & ?
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix}
= E
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix}.
\] (16)

Fill in the entries of the \(4 \times 4\) matrix.

5.9. Find the allowed values of \(E\).

**Problem 6.** List the references you used to help solve these problems.