Due Friday September 25

Problem 1. In class we argued that an ammonia atom in an electric field can be modeled by a two-level system, described by a Schrödinger equation

\[
\frac{i \hbar}{\partial t} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} E_0 + \epsilon & -\Delta \\ -\Delta & E_0 - \epsilon \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \tag{1}
\]

The constant \( E_0 \) plays no role in the dynamics: this is supposed to represent the energy when the Nitrogen atom is at one of its two potential minima. The term \( \Delta \) accounts for tunneling, and \( \epsilon \) is proportional to the electric field applied along the symmetry axis of the molecule.

Experiments show good agreement with this model, and the most accurate way to determine \( \epsilon \) and \( \Delta \) is not from a theoretical calculation, but by fitting the prediction of this model to experiment. One finds \( \Delta = 10^{-4} \text{eV} \), and \( \epsilon = \mu \mathcal{E} \), where \( \mathcal{E} \) is the electric field strength. The proportionality constant is the electric dipole moment, \( \mu = 1.6 \) debye. A debye is the CGS unit for electric dipole moments: \( 1 \text{ debye} = 3.3 \times 10^{-30} \text{Cm} \). A debye is a typical electric dipole moment for a molecule – it is equal to the charge of the electron times a distance of roughly 0.39 Bohr.

1.1. Derive an expression for the energy eigenvalues as a function of \( E_0 \), \( \Delta \), and \( \epsilon \). Make a sketch of Energy vs \( \epsilon \) for fixed \( E_0 \) and \( \Delta \). Please do not substitute in numbers – I just want to know the shape of the curve. [As an aside, you should see that one state is “high field seeking” – meaning its energy is lowest in regions of large field, while the other is “low field seeking”. This property is what is used for creating an inverted population for an ammonia Maser.]

1.2. Cavities vary greatly in the fields they contain. Some of the highest intensity microwave cavities are used in particle accelerators. For example, the next generation “Energy Recovering Linac” being built at Cornell has superconducting cavities with electric fields as large as 50 MV/m. For these extremely high-intensity cavities, how does \( \epsilon \) compare with \( \Delta \)? [More concretely, what is the ratio \( \epsilon/\Delta \)?] Would the perturbation theory we did in class be appropriate for these cavities? The original MASER experiments had much smaller fields.

Problem 2.

2.1. Two-level Hamiltonians, are often described in terms of \( 2 \times 2 \) matrices

\[
\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2-4}
\]
Are these Hermitian, Unitary, both?

2.2. What are the eigenvalues of the Hamiltonian

\[ H = E_0 \mathbf{1} + B_x \sigma_x + B_y \sigma_y + B_z \sigma_z, \quad (5) \]

where \( \mathbf{1} \) is the identity matrix. No need to find the eigenvectors, I just want the eigenvalues as a function of \( E_0 \) and the vector \( B_x, B_y, B_z \). This is a good result to remember (and I often calculate the eigenvalues of 2x2 matrices by writing them in this form).

Note: you don’t need to prove this, and it will not help you do the problem, but it is convenient to know that the eigenstates are of the form:

\[ \left( \begin{array}{c} \cos(\theta/2)e^{i\phi/2} \\ \sin(\theta/2)e^{-i\phi/2} \end{array} \right), \quad (6) \]

where \( \tan \theta = \sqrt{B_x^2 + B_y^2}/B_z \) and \( \tan \phi = B_y/B_x \) describe the direction of the vector \( \mathbf{B} \).

**Problem 3.** In the next problem we will use the following integral

\[ I = \int dx \frac{\sin^2 x}{x^2} = \pi. \quad (7) \]

Here you will verify that result using a trick called “integrating under the integral.” [Apparently Feynman was fond of tricks like this.]

3.1. By integrating over \( t_1 \) and \( t_2 \), show

\[ I = \frac{1}{4} \int_\infty^{-\infty} dx \int_{-1}^{1} dt_1 \int_{-1}^{1} dt_2 e^{ix(t_1+t_2)}. \quad (8) \]

3.2. Perform the \( x \) integral in EQ. (8), using the identity

\[ \int_{-\infty}^{\infty} dx e^{i sx} = 2\pi \delta(s). \quad (9) \]

3.3. Now the \( t_1 \) and \( t_2 \) integrals should now be straightforward. Perform them.

**Problem 4.** Although we did not spend much time exploring it, in class we showed that the probability of an ammonia atom absorbing radiation of detuning \( \delta \) in time \( t \) is

\[ P = 4\epsilon^2 t^2 \frac{\sin^2(\delta t/2)}{(\delta t/2)^2}, \quad (10) \]

where \( \epsilon \) is proportional to the electric field (so \( \epsilon^2 \) is proportional to the intensity of the light, \( I_0 \)). Imagine we have many ammonia atoms, and light of many frequencies. The total energy absorbed from the light in that time is then

\[ E \propto I_0 t^2 \int d\delta \frac{\sin^2(\delta t/2)}{(\delta t/2)^2}. \quad (11) \]
Use the identity
\[
\int dx \frac{\sin^2 x}{x^2} = \pi
\]
(12)
to show that the absorbed power is a constant, independent of time.

**Problem 5.** Here we will use dimensional analysis to estimate the ionization energy of hydrogen. The various quantities which come to mind as being involved are: The ionization energy \( E_0 \), the proton mass \( m_p \), electron mass \( m_e \), Coulomb’s constant \( k = e^2/(4\pi\epsilon_0) \), planck’s constant \( h \) (or equivalently \( \hbar = h/2\pi \)). The physics is non-relativistic, so we don’t expect to need \( c \).

5.1. How many independent dimensionless quantities can you construct from these parameters? Write them down.

5.2. What is the most general expression for \( E_0 \) which only involves these scales, and is dimensionally consistent.

5.3. If we assume that \( m_p \) drops out, what do you find for the ionization energy? [Note, that as with any other dimensional argument, all you get is an order of magnitude.] Give a number in eV.

**Problem 6.** The Schrodinger equation for a simple harmonic oscillator reads
\[
-\frac{\hbar^2}{2m} \partial_x^2 \psi(x) + \frac{1}{2} m \omega^2 x^2 \psi(x) = E \psi(x).
\]
(13)
Rescale the variables so that \( y = x/x_0 \) and \( E = E_0 / E_0 \) to reduce this to the dimensionless equation
\[
-\frac{1}{2} \partial_y^2 \psi + \frac{1}{2} y^2 \psi = E \psi.
\]
(14)
What is \( x_0 \) and \( E_0 \)?

**Problem 7.** A typical model for the potential between two atoms is
\[
V = \frac{\alpha}{x^{12}} - \frac{\beta}{x^6},
\]
where \( \alpha \) and \( \beta \) are constants which depend on the atom. This is often known as a “6-12” potential.

7.1. Sketch this potential.

7.2. What are the units of \( \alpha \) and \( \beta \)?

7.3. There is a unique length-scale \( r_0 \) you can make out of \( \alpha \) and \( \beta \). What is \( r_0 \)?

7.4. Since there is only one length-scale here, the location of the minimum of \( V \), must roughly be given by \( r_0 \). Use calculus to find the location of the minimum \( r^* \), and calculate \( r^*/r_0 \).

**Problem 8. Feedback**

8.1. How long did this homework take?

8.2. Which of the following words come to mind when you think about this homework (feel free to add your own words if you have something better): frustrating, fun, tedious, insightful, hard, easy, useful, useless, fair, unfair