Problem 1. Landau Levels
Here we will find the energy eigenstates a quantum mechanical particle moving in two dimensions in a uniform magnetic field. That is we will consider the Schrödinger equation

\[ E\psi = -\frac{\hbar^2}{2m} (\nabla - iqA)^2 \psi, \tag{1} \]

where \( \nabla \times A = B \). We will take \( B \) to point in the \( \hat{z} \) direction, and be uniform in space. The vector potential is not unique, we can add a divergence of any function to it. This is often described as “gauge freedom.”

1.1. Show that

\[ A = \Lambda x \hat{y} \]

, where \( \Lambda \) is a constant, corresponds to a uniform magnetic field. This is known as the Landau gauge, and it is particularly useful for this calculation.

1.2. Within the Landau gauge, the Hamiltonian involves \( \partial_x, \partial_y, \) and \( x \). It does not depend explicitly on \( y \), so the momentum in the \( \hat{y} \) direction is a good quantum number. Thus we write

\[ \psi(x, y) = e^{ik_y y} \phi_{k_y}(x). \tag{2} \]

Plug this into Eq. (1) and get an equation for \( \phi_{k_y}(x) \).

1.3. You should recognize that this equation is the same equation obeyed by the wave function for a 1D quantum harmonic oscillator. What is the frequency \( \omega_0 \) of the oscillator? Thinking about the classical motion of a charged particle in a magnetic field, what is the physical significance of this frequency?

1.4. From your knowledge of the harmonic oscillator eigen-energies, what are the energies of all the single particle states? Note: since \( E \) does not depend on \( k_y \), you have discovered a massive degeneracy.

Problem 2. Guiding Centers
Consider a charged particle moving in 2D in a uniform magnetic field within the Landau Gauge:

\[ H = \frac{1}{2m} \left[ p_x^2 + (p_y - qBx)^2 \right]. \tag{3} \]

Use Heisenberg equations of motion to calculate the rate of change of these four quantities: \( X = \langle \hat{x} \rangle \), \( Y = \langle \hat{y} \rangle \), \( P_x = \langle p_x \rangle \), and \( P_y = \langle \hat{p}_y - qB\hat{x} \rangle \). This should yield a set of four coupled differential equations which you can solve.
Hint 1: Start with the equations of motion for $\Pi_x$ and $\Pi_y$. These will be closed. Solve them. Substitute the solutions into the equations for $X$ and $Y$. Don’t forget your constants of integration. 

Hint 2: The Heisenberg equations of motion read:

\[
\frac{\partial \langle \hat{O} \rangle (t)}{\partial t} = \frac{1}{i\hbar} \int dx \psi^\ast(x, t)(\hat{O}\hat{H} - \hat{H}\hat{O})\psi(x, t) \tag{4}
\]

\[
= \frac{1}{i\hbar} \langle [\hat{O}, \hat{H}] \rangle. \tag{5}
\]

Hint 3: In case 2D confuses you – recall $x$ commutes with $y$ and $p_y$.

Hint 4: Recall the following:

\[
[x, \hat{p}] = i\hbar \tag{6}
\]

\[
[x, \hat{p}^2] = 2i\hbar \hat{p} \tag{7}
\]

\[
[x, x] = 0 \tag{8}
\]

\[
[x, x^2] = 0 \tag{9}
\]

\[
[\hat{p}, x] = -i \tag{10}
\]

\[
[\hat{p}, x^2] = -2i\hbar \hat{x} \tag{11}
\]

\[
[\hat{p}, \hat{p}] = 0 \tag{12}
\]

\[
[\hat{p}, \hat{p}^2] = 0 \tag{13}
\]

Problem 3. Maxwell’s Equations

We define the electromagnetic field tensor as

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{14}
\]

where $\mu, \nu = t, x, y, z$, and $A_t = -\phi$ is the scalar potential, and $A_x, A_y, A_z$ are the components of the vector potential.

3.1. What are the following in terms of $E$ and $B$: $F_{xt}, F_{yt}, F_{zt}, F_{xy}, F_{yz}, F_{zx}$?

3.2. Consider the quantity

\[
\Lambda_{xyz} = \partial_x F_{yz} + \partial_y F_{zx} + \partial_z F_{xy}. \tag{15}
\]

By writing $F$ in terms of $A$, show that $\Lambda_{xyz} = 0$. This should be part of one of Maxwell’s equations. Which one is it?

3.3. Consider the quantity

\[
\Lambda_{txy} = \partial_t F_{xy} + \partial_x F_{yt} + \partial_y F_{tx}. \tag{16}
\]

By writing $F$ in terms of $A$, show that $\Lambda_{txy} = 0$. This should be part of one of Maxwell’s equations. You can complete that equation by looking at $\Lambda_{tyz}$ and $\Lambda_{tz}$ (but you do not need to).
We will not prove it here, but it turns out that in this notation the other two Maxwell equations are

\[
\begin{align*}
\partial_x F_{xt} + \partial_y F_{yt} + \partial_z F_{zt} &= 0 \quad (17) \\
-\partial_t F_{tx} + \partial_y F_{yx} + \partial_z F_{zx} &= 0 \quad (18) \\
-\partial_t F_{ty} + \partial_x F_{xy} + \partial_z F_{zy} &= 0 \quad (19) \\
-\partial_t F_{tz} + \partial_x F_{xz} + \partial_y F_{yz} &= 0 \quad (20)
\end{align*}
\]

**Problem 4. Goldstone mode**

Consider the following wave equation

\[
\partial_t^2 \phi - \partial_x^2 \phi - m^2 \phi + \lambda^2 |\phi|^2 \phi = 0. \quad (22)
\]

This is *not* a gauge theory.

**4.1.** Linearize Eq. (22) about \( \phi = \phi_0 = m/\lambda \).

**4.2.** What are the spectra of excitations? You should find one massive (gapped) mode. You will also find a gapless mode – the Goldstone mode.

This is a generic result. Whenever you have a continuously degenerate ground state, you will have a gapless mode. For example, in a crystal there is a continuous degeneracy corresponding to where the first atom sits. Once you specify its location you know where all of them are. Thus a crystal should have a gapless mode. It in fact has 3 gapless modes: the two transverse acoustic phonons, and a single longitudinal acoustic phonon.

**Problem 5. Feedback**

**5.1.** How long did this homework take?

**5.2.** Which of the following words come to mind when you think about this homework (feel free to add your own words if you have something better): frustrating, fun, tedious, insightful, hard, easy, useful, useless, fair, unfair