Problem 1. Wavefunctions – 4 points

1.1. Write a two-particle wavefunction which is bosonic.

Solution 1.1. 1 point
There are many different solutions. For example
\[
\psi(x_1, x_2) = 1
\]
Anything works as long as \( \psi(x_1, x_2) = \psi(x_2, x_1) \).

1.2. Write a two-particle wavefunction which is fermionic.

Solution 1.2. 1 point
There are many different solutions. For example
\[
\psi(x_1, x_2) = x_1 - x_2
\]
Anything works as long as \( \psi(x_1, x_2) = -\psi(x_2, x_1) \).

1.3. Write a two-particle wavefunction which is neither bosonic nor fermionic.

Solution 1.3. 1 point
There are many different solutions. For example
\[
\psi(x_1, x_2) = x_1
\]

1.4. Write a two-particle wavefunction in which the two particles are bound closely together, but their center of mass is delocalized.

Solution 1.4. 1 point
There are many different solutions. For example
\[
\psi(x_1, x_2) = e^{-\frac{(x_1 - x_2)^2}{\xi^2}},
\]
where \( \xi \) controls how tightly they are bound.
Problem 2. Neutrino Oscillations – 13 points

The 2015 Nobel Prize in Physics was awarded jointly to Takaaki Kajita and Arthur B. McDonald “for the discovery of neutrino oscillations, which show that neutrinos have mass.” This year’s Nobel lectures will be webcast on December 8 at http://www.kva.se/sv/Kalendariumlista/2015/the-nobel-lectures-2015/.

There are 3 flavors of neutrino: electron, muon and tau. The weak interaction produces neutrinos in these flavor eigenstates, $|e\rangle, |\mu\rangle, |\tau\rangle$, which can be represented as column vectors in flavor space:

$$|e\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$ \hfill (1)

$$|\mu\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$ \hfill (2)

$$|\tau\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$ \hfill (3)

These are not eigenstates of the Hamiltonian. One model, consistent with the experimental data, is that the energy eigenstates are:

$$|1\rangle = \begin{pmatrix} 0.82 \\ -0.51 \\ 0.26 \end{pmatrix}$$ \hfill (4)

$$|2\rangle = \begin{pmatrix} 0.55 \\ 0.58 \\ -0.60 \end{pmatrix}$$ \hfill (5)

$$|3\rangle = \begin{pmatrix} 0.16 \\ 0.63 \\ 0.75 \end{pmatrix}$$ \hfill (6)

with energies $E_1 = \sqrt{(5\text{meV})^2 + p^2c^2}$, $E_2 = \sqrt{(10\text{meV})^2 + p^2c^2}$, and $E_3 = \sqrt{(50\text{meV})^2 + p^2c^2}$, where $p$ is the neutrino momentum. We will assume $p = 10\text{MeV}/c$, a typical solar neutrino momentum.

In the sun, electron neutrino’s are produced.

2.1. 1 point In terms of a length 3 column vector, what is the flavor wavefunction $|\psi(t = 0)\rangle$ of a neutrino when it is first produced?
Solution 2.1. 1 point

\[ |\psi(t = 0)\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \]

2.2. 2 points In terms of the energy eigenstates, this wavefunction can be written \( |\psi(t = 0)\rangle = \alpha|1\rangle + \beta|2\rangle + \gamma|3\rangle \). Find \( \alpha, \beta, \gamma \).

Solution 2.2. 2 points

Setting up the problem – 1 pt
This is a linear algebra problem
\[
\begin{pmatrix}
0.82 & 0.55 & 0.16 \\
-0.51 & 0.58 & 0.63 \\
0.26 & -0.60 & 0.75 \\
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\end{pmatrix} =
\begin{pmatrix}
1 \\
0 \\
0 \\
\end{pmatrix}
\]

Solving the linear algebra problem – 1 pt
The easiest (but hardly the only) way to solve this problem is to recognize that the matrix is Unitary, so that
\[
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\end{pmatrix} =
\begin{pmatrix}
0.82 & -0.51 & 0.26 \\
0.55 & 0.58 & -0.60 \\
0.16 & 0.63 & 0.75 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0 \\
\end{pmatrix} =
\begin{pmatrix}
0.82 \\
0.55 \\
0.16 \\
\end{pmatrix}
\]

2.3. 2 points What is the flavor wavefunction at some later time \( t \), \( |\psi(t)\rangle \). Write your solution both as a sum of terms involving the energy eigenstates, and in the form of a length 3 column vector.

Solution 2.3. 2 points

Expression in terms of energy eigenstates – 1 point

\[
|\psi(t)\rangle = \alpha e^{-iE_1t/h}|1\rangle + \beta e^{-iE_2t/h}|2\rangle + \gamma e^{-iE_3t/h} \\
= 0.82e^{-iE_1t/h}|1\rangle + 0.55e^{-iE_2t/h}|2\rangle + 0.16e^{-iE_3t/h}|3\rangle
\]
Expression in terms of a length 3 column vector – 1 point

\[ |\psi(t)\rangle = \begin{pmatrix}
0.67e^{-iE_1t/\hbar} + 0.30e^{-iE_2t/\hbar} + 0.03e^{-iE_3t/\hbar} \\
-0.42e^{-iE_1t/\hbar} + 0.319e^{-iE_2t/\hbar} + 0.10e^{-iE_3t/\hbar} \\
0.21e^{-iE_1t/\hbar} - 0.33e^{-iE_2t/\hbar} + 0.12e^{-iE_3t/\hbar}
\end{pmatrix} \]

2.4. 1 point Assuming that the neutrinos move at the speed of light, how long do they take to travel from the sun to the earth?

Solution 2.4. 1 point
It is roughly \( d = 1.5 \times 10^{11} \text{ m} \) between the earth and the sun. The transit time is then

\[ t = \frac{d}{c} = \frac{1.5 \times 10^{11} \text{ m}}{3 \times 10^8 \text{ m/s}} = 500 \text{ s} \]

2.5. 1 point Calculate \( E_1t/\hbar \).

Solution 2.5. 1 point

\[ E_1t/\hbar = \frac{(10^7 \text{ eV})(500 \text{ s})}{6.6 \times 10^{-16} \text{ eV s}} = 7.6 \times 10^{24}. \]

2.6. 1 point Calculate \( \phi_{21} = (E_2 - E_1)t/\hbar \). [Note: The easiest way to calculate this is to Taylor expand \( E_2 \) and \( E_1 \), taking \( pc \) large.]

Solution 2.6. 1 point
Following the suggestion

\[ E_2 = pc + \frac{(5 \text{ meV})^2}{2pc} + \cdots = pc + 1.25 \times 10^{-12} \text{ eV} + \cdots \]

and

\[ E_1 = pc + \frac{(10 \text{ meV})^2}{2pc} + \cdots = pc + 5 \times 10^{-12} \text{ eV} + \cdots \]

The phase factor is then

\[ \phi_{21} = (E_2 - E_1)t/\hbar = \frac{(3.75 \times 10^{-12} \text{ eV})(500 \text{ s})}{6.6 \times 10^{-16} \text{ eV} \cdot \text{s}} = 2.8 \times 10^6. \]
2.7. 1 point Calculate $\phi_{31} = (E_3 - E_1)t/\hbar$. [Note: The easiest way to calculate this is to Taylor expand $E_3$ and $E_1$, taking $pc$ large.]

Solution 2.7. 1 point
Following the same procedure

$$E_3 = pc + \frac{(50 \text{meV})^2}{2pc} + \cdots = pc + 1.25 \times 10^{-10} eV + \cdots$$

The phase factor is then

$$\phi_{31} = (E_3 - E_1)t/\hbar = \frac{(1.24 \times 10^{-10} eV)(500 \text{s})}{6.6 \times 10^{-16} eV \cdot \text{s}} = 9.4 \times 10^8$$

2.8. The flavor wavefunction when the neutrinos hit the earth can be represented as a length 3 column vector of the form

$$|\psi(t)\rangle = e^{-iE_1t} \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

where $a$, $b$ and $c$ are complex numbers. How is the probability that a neutrino will be in an electron flavor eigenstate related to $a$, $b$, or $c$? Do not bother numerically calculating $a$, $b$, or $c$. Due to the large numbers, and round-off errors, you can’t expect it to be accurate enough to be worth the effort.

Solution 2.8. 1 point
The probability of being in an electron flavor wavefunction is $|a|^2$.

2.9. Suppose we have a reactor which produces electron neutrinos with momentum $p = 10 MeV/c$. They travel a distance of $d = 30 km$ before hitting a detector. Calculate the travel time $t$, the phases $\phi_{21}$, $\phi_{31}$, and the amplitudes $a$. Use these results to calculate the probability that the neutrinos are in an electron flavor eigenstate when they hit the detector.

Solution 2.9. 3 points
The travel time is

$$t = \frac{d}{c} = \frac{30 \times 10^3 m}{3 \times 10^8 m/s} = 10^{-4} s.$$ 

The phases are

$$\phi_{21} = (E_2 - E_1)t/\hbar = \frac{(3.75 \times 10^{-12} eV)(10^{-4} s)}{6.6 \times 10^{-16} eV \cdot s} = 0.57$$

$$\phi_{31} = (E_3 - E_1)t/\hbar = \frac{(1.24 \times 10^{-10} eV)(10^{-4} s)}{6.6 \times 10^{-16} eV \cdot s} = 18.8.$$
The amplitudes are

\[ a = 0.67 + 0.30e^{-i\phi_{21}} + 0.03e^{-i\phi_{31}} = 0.95 - 0.16i \]
\[ b = -0.42 + 0.319e^{-i\phi_{21}} + 0.10e^{-i\phi_{31}} = -0.05 + 0.17i \]
\[ c = 0.21 - 0.33e^{-i\phi_{21}} + 0.12e^{-i\phi_{31}} = -0.05 + 0.18i. \]

[Only the expression for \( a \) is needed.] The probability that the electrons are in an electron flavor eigenstate is

\[ P = |a|^2 = 0.93 \]

**Problem 3. Wave-nature of neutrons – 10 points**

Over the past few decades there have been a number of experiments which have observed the wave-nature of neutrons. [See for example, Nesvizhevsky et al. Nature 415, 297 (2002), available at http://www.nature.com/nature/journal/v415/n6869/abs/415297a.html.]

In a typical setup the neutrons are trapped by the gravitational potential and a reflecting surface. A reasonable model for the wavefunction of the neutron is a one-dimensional Schrödinger equation:

\[ i\hbar \partial_t \psi(z,t) = -\frac{\hbar^2}{2m} \partial^2_x \psi(z,t) + mgz\psi(z,t), \tag{8} \]

with boundary condition \( \psi(0,t) = 0 \). Here \( m \) is the neutron’s mass, and \( g \) is the acceleration due to gravity. One can adimensionalize this equation, so that it reads

\[ i\partial_\tau \psi(s,\tau) = -\frac{1}{2} \partial^2_s \psi(s,\tau) + s\psi(s,\tau), \tag{9} \]

where \( s = z/z_0 \) and \( \tau = t/t_0 \).

**3.1. 2 points** What are \( z_0 \) and \( t_0 \) in physical units (\( \mu \)m and s)? What is the characteristic energy scale of the system \( \epsilon_0 = \hbar/t_0 \) in peV?

**Solution 3.1. 2 points**

Making the substitutions,

\[ i\frac{\hbar}{t_0} \psi = -\frac{\hbar^2}{2mz_0^2} \partial^2_s \psi + s(mgz_0)\psi. \]

To be of the desired form

\[ \frac{\hbar}{t_0} = \frac{\hbar^2}{mz_0^2} = mgz_0. \]
The last equality gives
\[ z_0 = \left( \frac{\hbar^2}{m^2 g} \right)^{1/3}, \]
which can be substituted to yield
\[ t_0 = \frac{\hbar}{mg} \left( \frac{m^2 g}{\hbar^2} \right)^{1/3} = \left( \frac{\hbar}{mg^2} \right)^{1/3}. \]

### 3.2. Consider the variational wavefunction
\[ \psi(s) = A s e^{-\lambda s}, \quad (10) \]
where \( A \) and \( \lambda \) are parameters. Use the normalization condition on the wavefunction to determine \( A \).

#### Solution 3.2. 1 point
I will give full points for any measure you use – as long as you are consistent in the rest of the problem. Reasonable choices include
\[ \int ds |\psi(s)|^2 = 1 \]
\[ z_0 \int ds |\psi(s)|^2 = 1. \]
In the former case \( \psi \) is dimensionless, while in the latter it has units of \( 1/\sqrt{\text{length}} \) – the difference is if you adimensionalized \( \psi \). Regardless, you need to calculate
\[ \int_0^\infty |\psi(s)|^2 ds = |A|^2 \int_0^\infty s^2 e^{-2\lambda s} ds = \frac{|A|^2}{4\lambda^3}. \]
Using the simplest normalization this gives
\[ A = \sqrt{4\lambda^3}. \]

### 3.3. Use the variational principle and this wavefunction to estimate the ground state energy in units of \( \epsilon_0 \).

#### Solution 3.3. 3 points
Writing out the energy – 1 point
Using the measure appropriate for the dimensionless $\psi$, energy is

$$\frac{E}{\epsilon_0} = \int_0^\infty ds \left[ \frac{1}{2} |\partial_z \psi|^2 + s|\psi|^2 \right]$$

$$= 4\lambda^3 \int_0^\infty ds \left[ \frac{1}{2} e^{-2\lambda s} (1 - \lambda s)^2 + s^3 e^{-2\lambda s} \right]$$

$$= 4\lambda^3 \int_0^\infty ds e^{-2\lambda s} \left[ s^3 + \frac{1}{2} - \lambda s + \frac{1}{2} \lambda^2 s^2 \right].$$

Calculating the integrals – 1 point
We then use

$$\int_0^\infty e^{-\alpha s} ds = \frac{1}{\alpha}$$

$$\int_0^\infty s e^{-\alpha s} ds = \frac{1}{\alpha^2}$$

$$\int_0^\infty s^2 e^{-\alpha s} ds = \frac{2}{\alpha^3}$$

$$\int_0^\infty s^3 e^{-\alpha s} ds = \frac{6}{\alpha^4},$$

to get

$$\frac{E}{\epsilon_0} = 4\lambda^3 \left[ \frac{6}{(2\lambda)^4} + \frac{1}{2} \frac{1}{(2\lambda)^2} - \lambda \frac{1}{(2\lambda)^2} + \frac{\lambda^2}{2} \frac{2}{(2\lambda)^3} \right]$$

$$= \frac{1}{2} \left( \frac{3}{\lambda} + \lambda^2 \right).$$

Minimizing the energy – 1 point
We then minimize the energy,

$$\frac{\partial}{\partial \lambda} \frac{2E}{\epsilon_0} = -3 \frac{1}{\lambda^2} + 2\lambda = 0$$

This gives

$$\lambda = (3/2)^{1/3}.$$ 

Substituting this back into the expression for the energy yields

$$E = \frac{\epsilon_0}{2} \left( \frac{3}{(3/2)^{1/3}} + (3/2)^{2/3} \right)$$

$$= 1.97\epsilon_0.$$ 

3.4. Write a computer program to calculate the ground state energy in units of $\epsilon_0$. Give your
numerical answer here. Attach your program to the end of this exam. If you hand it in electronically, please be sure that your name is on it.

Solution 3.4. 4 points – 1 point for the correct answer, 3 for setting it up on the computer
I find numerically
\[ E = 1.85\epsilon_0 \]
My solutions are attached to the end.

Problem 4. How big is a neutron star? – 14 points We all know neutron stars are dense burned out stars. They are made from neutrons. It turns out that the only thing which is supporting them against gravitational collapse is the fact that neutrons are fermions.

4.1. 4 points Suppose the Neutron star contains \( N \) electrons, and has radius \( R \). The kinetic energy of the neutrons should be roughly the same as the kinetic energy of \( N \) neutrons in a 3D infinite square well of size \( R \times R \times R \). Use this argument to estimate the kinetic energy as a function of the radius of the star. Ignore relativistic effects, but include that fact that the neutrons are spin-1/2 fermions.

Solution 4.1. Since this is an estimate, there are many ways to do this. I asked for a particular approach (which is not necessarily the most accurate, but it is straightforward). Any other approach which gets the right \( N \) and \( R \) scaling will get 3 points. An approach which gets the right \( R \) scaling, but not the right \( N \) scaling will get 2 points.

Square well approach – 4 points
In a box of size \( R \times R \times R \), the eigenstates have energy
\[ E_{n_x,n_y,n_z} = \frac{\hbar^2 (2\pi)^2}{2mR^2} (n_x^2 + n_y^2 + n_z^2), \]
where \( n_x, n_y, n_z \) are positive integers. We fill all states up to the Fermi energy \( \epsilon_f \), whence
\[ N = 2 \sum_{n_x,n_y,n_z} \theta(\epsilon_f - E_{n_x,n_y,n_z}), \]
where \( \theta \) is the Heaviside step function, which is zero for negative arguments, and 1 for positive arguments. The two comes from spin. Replacing the sum with an integral, we get
\[ N = 2 \frac{1}{8} \int dx \, dy \, dz, \]
\[ x^2 + y^2 + z^2 < 2mR^2 \epsilon_f / (4\pi^2 \hbar^2) \]
where the factor of $1/8$ comes from restricting to positive $n$. Changing to spherical coordinates

\[
N = \frac{1}{4} 4\pi \int_0^{2mR^2\epsilon_f/(4\pi^2\hbar^2)} r^2 dr
= \frac{\pi}{3} \left( \frac{2mR^2 \epsilon_f}{4\pi^2\hbar^2} \right)^3.
\]

The kinetic energy is

\[
E = 2 \sum_{n_x,n_y,n_z} E_{n_x,n_y,n_z} \theta(\epsilon_f - E_{n_x,n_y,n_z}).
\]

We again replace this with an integral,

\[
E = 2 \frac{1}{8} \int dx \, dy \, dz \, \frac{\hbar^2 (2\pi)^2}{2mR^2} (x^2 + y^2 + z^2).
\]

Changing to spherical coordinates,

\[
E = \frac{1}{4} 4\pi \frac{\hbar^2 (2\pi)^2}{2mR^2} \int_0^{2mR^2\epsilon_f/(4\pi^2\hbar^2)} r^4 dr
= \frac{\pi}{5} \left( \frac{2mR^2 \epsilon_f}{4\pi^2\hbar^2} \right)^5
= \frac{\pi}{5} \left( \frac{2mR^2 \epsilon_f}{4\pi^2\hbar^2} \right)^5 \left( \frac{N}{4\pi} \right)^5
= \frac{2\hbar^2 \pi^4/3}{5mR^2} N^{5/3}.
\]

4.2. Estimate the gravitational potential energy of $N$ neutrons forming a ball of radius $R$. Make any approximations that you feel appropriate – just be explicit in your approximation.

**Solution 4.2. 3 points** Any valid solutions gets full marks. You must get the $N$ and $R$ scaling right, and be explicit in your approximations.

The simplest estimate is simply to use dimensional analysis. The relevant mass in the problem is the total mass of the star. The only radius in the problem is $R$, so the gravitational potential energy must be

\[
E = -G \frac{(Nm)^2}{R}.
\]

A more sophisticated solution is to treat the star as a uniform spherical shell of radius $R$ and mass $Nm$. We imagine slowly adding mass to the shell. When it has mass $\mu$, the energy to
add mass \( d\mu \) is
\[
dE = -G\frac{\mu d\mu}{R}.
\]
The total energy is then
\[
E = \frac{G}{R} \int_0^{Nm} \mu d\mu = -G\frac{(Nm)^2}{2R}.
\]
(12)

An even more sophisticated solution is to treat the star as a uniform mass of density \( \rho = \frac{3Nm}{4\pi R^3} \). We build it up from spherical shells. When it is of radius \( r \) we add a new shell of thickness \( dr \). The energy to add this shell is
\[
dE = -G\frac{MdM}{r} = -G\left(4\pi r^3 \rho\right)\left(4\pi r^2 \rho dr\right)
\]
\[
= -3G(Nm)^2 \frac{r^4}{R^6} dr.
\]
The total gravitational energy is then
\[
E = -3G(Nm)^2 \int_0^R \frac{r^4}{R^6} dr = -\frac{3}{5}G\frac{(Nm)^2}{R}.
\]
(13)

### 4.3. Sketch the total energy of the neutron star as a function of radius. Label your axes.

**Solution 4.3. 3 points**

Our expression for the energy is
\[
E = \frac{2h^2 \pi^{4/3}}{5mR^2} N^{5/3} - \frac{3}{5}G\frac{(Nm)^2}{R}.
\]
The constants are not so important, and this is of the form
\[
E = \alpha \frac{N^{5/3}}{R^2} - \beta \frac{N^2}{R},
\]
with
\[
\alpha = \frac{2h^2 \pi^{4/3}}{5m},
\]
\[
\beta = \frac{3}{5}Gm^2
\]

You will get full marks for something like
If the shape is wrong you get zero points. If the shape is right, but you have a crazy scale, so that you can not see any features, you get 1 point. If you forget to put labels, you lose a point.

4.4. Minimize the energy to derive a relationship between the radius of a neutron star and its mass.

Solution 4.4. 2 points We use our expression

\[ E = \alpha \frac{N^{5/3}}{R^2} - \beta \frac{N^2}{R}. \]

Minimizing with respect to \( R \) gives

\[ \frac{\partial E}{\partial R} = -2\alpha \frac{N^{5/3}}{R^3} + \beta \frac{N^2}{R^2} = 0 \]

or

\[ R = \frac{2\alpha}{\beta N^{1/3}}. \]

Plugging in our expressions for \( \alpha \) and \( \beta \) gives

\[ R = \frac{4\hbar^2 \pi^{4/3}}{3Gm^3 N^{1/3}}. \]

4.5. A typical neutron star has a mass of about two solar masses. Using your size estimate, how large will this neutron star be? How does that compare to the size of the sun?
Solution 4.5. 1 point The sun weighs $M_\odot = 2 \times 10^{30}$ kg. The number of neutrons is

$$N = 2M_\odot/m = \frac{4 \times 10^{30} \text{kg}}{1.7 \times 10^{-27} \text{kg}} = 2.4 \times 10^{57}$$

yielding

$$R = \frac{20h^2\pi^{4/3}}{15Gm^3N^{1/3}}$$

$$= \frac{4(1.05 \times 10^{-34} \text{m}^2\text{kg/s})^2\pi^{4/3}}{3(6.7 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2})(1.7 \times 10^{-27} \text{kg})^3(2.4 \times 10^{57})^{1/3}}$$

$$= 1.5 \times 10^{4} \text{m.}$$

For comparison, the radius of the sun is $R_\odot = 6.9 \times 10^8 \text{m}$: 4 orders of magnitude larger.

4.6. A typical nucleus has size $r = 10^{-15} \text{m}$ – a lengthscale known as a “Fermi.” How does the density of a neutron star compare with that of a nucleus?

Solution 4.6. 1 point

The number density of our Neutron star is

$$\rho = \frac{N}{R^3} = \frac{2.4 \times 10^{57}}{(1.5 \times 10^4 \text{m})^3} = 8.7 \times 10^4 \text{m}^{-3}.$$  

A typical nuclear density is

$$\rho_n = 10^{45} \text{m}^{-3}.$$  

These are surprisingly similar. [Which probably means that inter-nuclear forces are starting to become important in the center of a neutron star.]

Problem 5. Hyperfine Structure – 9 points You know a lot about the electronic excitations of hydrogen (and other atoms). One thing which is probably less familiar to you is the fact that the electronic spin can couple to the motion of the electron (so-called “spin-orbit coupling”). Such spin-orbit coupling is responsible for the small splitting between the lines in sodium. This splitting is so small that it is referred to as “fine structure”.

Here we are going to explore an even more subtle effect – namely that there is a coupling between the electronic and nuclear spin in hydrogen – leading to very-very-very small splittings of spectral lines – so-called hyper-fine structure. This splitting is in the microwave frequency band ($\nu = 1.4$ GHz, or $\lambda = 21 \text{cm}$). It is very important for astrophysics, as you can use it to detect clouds of hydrogen.
The electron and the proton in the hydrogen atom can each be in one of two states: giving a total of four possibilities:

1. \( |↑↑⟩ \): electron up, proton up
2. \( |↑↓⟩ \): electron up, proton down
3. \( |↓↑⟩ \): electron down, proton up
4. \( |↓↓⟩ \): electron down, proton down

We will let \( S_e \) and \( S_p \) be the spin operators for the electron and the proton.

5.1. What is \( (S_e)_z |↑↓⟩ \)?

**Solution 5.1. 1 point**

\[
(S_e)_z |↑↓⟩ = \frac{\hbar}{2} |↑↓⟩
\]

5.2. What is \( (S_e)_x |↑↓⟩ \)?

**Solution 5.2. 1 point**

\[
(S_e)_x |↑↓⟩ = \frac{\hbar}{2} |↓↓⟩
\]

5.3. What is \( (S_e)_z (S_p)_z |↑↓⟩ \)?

**Solution 5.3. 1 point**

\[
(S_e)_z (S_p)_z |↑↓⟩ = -\frac{\hbar^2}{4} |↑↓⟩
\]

5.4. By symmetry, the coupling between the electron spin and the proton spin takes the form

\[
H = A \left[(S_e)_z (S_p)_z + (S_e)_y (S_p)_y + (S_e)_x (S_p)_x \right].
\]  
(15)

Acting on one of the wavefunctions,

\[
H |↑↑⟩ = a |↑↑⟩ + b |↑↓⟩ + c |↓↑⟩ + d |↓↓⟩.
\]  
(16)

Find \( a, b, c, d \).
Solution 5.4. 1 point The easiest way to do this problem (and the next few) is to recall from one of our homeworks that

\[(S_e)_y(S_p)_y + (S_e)_x(S_p)_x = \frac{1}{2} [(S_e)_+(S_p)_- + (S_e)_-(S_p)_+] \]

in which case we instantly see that \(a = A(\bar{h}/2)^2, b = 0, c = 0, d = 0\).

The longer way to do this is to note

\[(S_e)_z(S_p)_z|\uparrow\uparrow\rangle = (\bar{h}/2)|\uparrow\uparrow\rangle \]

\[(S_e)_x(S_p)_x|\uparrow\uparrow\rangle = (\bar{h}/2)|\downarrow\downarrow\rangle \]

\[(S_e)_y(S_p)_y|\uparrow\uparrow\rangle = -(\bar{h}/2)|\downarrow\uparrow\rangle. \]

Putting this together, we then have \(a = A\bar{h}^2/4, b = 0, c = 0, d = 0\).

5.5.

\[H|\uparrow\downarrow\rangle = e|\uparrow\uparrow\rangle + f|\uparrow\downarrow\rangle + g|\downarrow\uparrow\rangle + h|\downarrow\downarrow\rangle. \tag{17} \]

Find \(e, f, g, h\).

Solution 5.5. 1 point The easy way to do this is to use

\[H|\uparrow\downarrow\rangle = A(S_e)_z(S_p)_z|\uparrow\downarrow\rangle + \frac{A}{2}(S_e)_+(S_p)_-|\uparrow\downarrow\rangle + \frac{A}{2}(S_e)_-(S_p)_+|\uparrow\downarrow\rangle \]

\[= -A(\bar{h}/2)|\uparrow\downarrow\rangle + (A/2)\bar{h}^2|\downarrow\uparrow\rangle. \]

Thus \(e = 0, f = -A\bar{h}^2/4, g = (A/2)\bar{h}^2, h = 0\).

The other approach is to note

\[(S_e)_z(S_p)_z|\uparrow\downarrow\rangle = -(\bar{h}^2/4)|\uparrow\downarrow\rangle \]

\[(S_e)_x(S_p)_x|\uparrow\downarrow\rangle = (\bar{h}^2/4)|\downarrow\uparrow\rangle \]

\[(S_e)_y(S_p)_y|\uparrow\downarrow\rangle = (\bar{h}^2/4)|\downarrow\uparrow\rangle. \]

Putting these together again gives \(e = 0, f = -A\bar{h}^2/4, g = A\bar{h}^2/2, h = 0\).

5.6.

\[H|\downarrow\uparrow\rangle = i|\uparrow\uparrow\rangle + j|\uparrow\downarrow\rangle + k|\downarrow\uparrow\rangle + l|\downarrow\downarrow\rangle. \tag{18} \]

Find \(i, j, k, l\).

Solution 5.6. 1 point By symmetry we have \(i = h = 0, j = g = A\bar{h}^2/2, k = f = -A\bar{h}^2/4, 0\).
5.7.

\[ H|\downarrow\downarrow\rangle = m|\uparrow\uparrow\rangle + n|\uparrow\downarrow\rangle + o|\downarrow\uparrow\rangle + p|\downarrow\downarrow\rangle. \] (19)

Find \( m, n, o, p. \)

**Solution 5.7. 1 point** By symmetry we have \( m = d = 0, n = c = 0, o = b = 0, p = a = Ah^2/4. \)

5.8. Suppose \(|\psi\rangle = \alpha|\uparrow\uparrow\rangle + \beta|\uparrow\downarrow\rangle + \gamma|\downarrow\uparrow\rangle + \delta|\downarrow\downarrow\rangle\) is an energy eigenstate, with energy \( E. \) Show that the coefficients \( \alpha, \beta, \gamma, \delta \) obey an equation

\[
\begin{pmatrix}
? & ? & ? & ?
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix} = E
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix}.
\] (20)

Fill in the entries of the \( 4 \times 4 \) matrix.

**Solution 5.8. 1 point** The matrix is

\[
\begin{pmatrix}
a & b & c & d \\
e & f & g & h \\
i & j & k & l \\
m & n & o & p
\end{pmatrix} = \frac{Ah^2}{4}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 \\
0 & 2 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

5.9. Find the allowed values of \( E. \)

**Solution 5.9. 1 point** The eigenvalues are \( Ah^2/4, Ah^2/4, Ah^2/4, -3Ah^2/4. \)

**Problem 6.** List the references you used to help solve these problems.
Constructing the Hamiltonian

First we make a function which generates the finite difference approximation to the second derivative.

```python
In [2]: def SecondDerivMatrix(numpoints, dx, periodic):
   """SecondDerivMatrix(numpoints, dx, periodic) returns a sparse matrix
   which represents the second derivative -- using a 3-point derivative.
   It takes periodic needed arguments:
   numpoints -- an integer which is how many points are in the grid
   dx -- the real space lattice spacing
   periodic -- do we use periodic boundary conditions?
   Setting periodic to "False" will give "hard wall boundaries"
   """
   unitlist=ones(numpoints)  # just a list of 1's whose length is equal to the length of the grid
   if periodic:
       return dia_matrix((
       (unitlist/(dx**2),unitlist/(dx**2),-2*unitlist/(dx**2),unitlist/(dx**2),
       (1,-1,0,numpoints-1,1-numpoints)),  # this line specifies which the diagonals
       shape=(numpoints,numpoints))
   else:
       return dia_matrix((
       (unitlist/(dx**2),unitlist/(dx**2),-2*unitlist/(dx**2)),
       (1,-1,0),  # this line specifies which the diagonals
       shape=(numpoints,numpoints))
```
Next we make the potential

In [3]: # Hit shift-enter
def PotentialMatrix(potentialfunction, minx, maxx, numpoints=None, dx=None):
    """PotentialMatrix(potentialfunction, minx, maxx, periodic, numpoints, dx)
    generates a finite difference approximation to the operator V(x).
    It is called by the following arguments:
    potentialfunction -- a function which when called with x returns V(x)
    minx -- smallest x in grid
    maxx -- largest x in grid
    numpoints -- number of points in grid
    dx -- grid spacing
    Either specify numpoints or dx -- but not both""
    
    # First we check to see if numpoints and/or dx are specified
    # #If you are new to programming, don't worry too much about
    # #this section -- it is just here to give us the flexibility
    # #to specify the grid either by the number of points or
    # #the spacing.
    # # in the former case we would use linspace(xmin,xmax,numpoints)
    # in the latter case we would use arange(xmin,xmax,dx)
    #
    # It is useful to see this sort of construction, as undoubtedly
    # you will want to throw error messages some time in the future.
    #
    if numpoints==None: #was numpoints specified
        if dx==None: #was dx specified
            # oops -- neither are specified, give error message
            raise Exception("Error: you must specify either numpoints
or dx")
        else: # great -- dx is specified, but numpoints is not
            grid = arange(minx,maxx+dx,dx)
    elif dx==None: # check to make sure dx is not specified
        grid=linspace(minx,maxx,numpoints)
    else: # both are specified
        if dx*(numpoints-1)==(maxx-minx): # are they consistent?
            grid=linspace(minx,maxx,numpoints)
        else:
            raise Exception("error: dx and numpoints are not consi-
stent -- please just specify one of them")
    
    # now generate the values of the potential on the grid
    potvals=array([potentialfunction(x) for x in grid])
Finally we make a function which generates the Hamiltonian as a matrix

In [4]: def HamiltonianMatrix(potentialfunction, minx, maxx, numpoints, periodic, hbar=1, mass=1):
    """HamiltonianMatrix(potentialfunction, minx, maxx, numpoints, hbar=1, mass=1)
    generates a finite difference approximation to the hamiltonian operator \( H \)
    for the case of a single particle in a potential \( V(x) \)
    It is called by the following required arguments:
    potentialfunction -- a function which when called with \( x \) returns \( V(x) \)
    minx -- smallest \( x \) in grid
    maxx -- largest \( x \) in grid
    numpoints -- number of points in grid
    It also has two optional arguments, which are assumed to be equal to unity if they
    are not specified
    hbar -- Planck's constant divided by 2 pi
    mass -- the particle mass
    """
    dx=1.*(maxx-minx)/(numpoints-1) # calculate grid spacing
    kin=-(0.5*hbar**2/mass)*SecondDerivMatrix(numpoints=numpoints, dx=dx, periodic=periodic) # calculate kinetic energy operator
    pot=PotentialMatrix(potentialfunction=potentialfunction, minx=minx, maxx=maxx, numpoints=numpoints) #calculate potential energy operator
    return kin+pot

Eigenvalues of gravitational potential

In [5]: def linfun(x):
    return x

In [16]: haml=HamiltonianMatrix(potentialfunction=linfun, minx=0, maxx=10, numpoints=1000, periodic=False)

In [7]: from scipy.sparse.linalg import eigsh # load the function which calculates eigenvectors of hermitian matrices
Comparison with variational result

This section is not required -- I just wanted to check how things look

In [18]: `energies, wavefunctions = eigsh(ham1, k=10, sigma=0, return_eigenvectors=True)`

In [24]: `def trial(x):
   return x * exp(-x*(2/3)**(1/3))`

In [25]: `grid = linspace(0, 10, 1000)
trialvals = trial(grid)`

In [32]: `plot(grid, wavefunctions[:, 0])
plot(grid, 0.22 * trialvals)
title("Lowest eigenstate")
xlabel("x/d")
ylabel("$\psi$ [Arbitrary Units]")`

Out[32]: `<matplotlib.text.Text at 0x10e723490>`
We see that the main difference is in the tail: Our ansatz does not decay as fast as the exact wavefunction (which asymptotically is \( \psi \sim e^{-x^{3/2}} \)). Regardless, the energy was not too bad.

**Visualizing wavefunctions**

Again, this part is not necessary.

In [35]:

```python
potvalues = grid
```

In [39]:

```python
plot(grid, potvalues, linewidth=4, color=(0, 0, 0))  # plot the parabola representing the harmonic potential
ylim(0, 10)  # rescale the y-axis
xlabel("x/d")  # label the x-axis
ylabel("Energy/$\epsilon_0$")  # label the y-axis
title("Eigenstates of gravitational potential, shifted by energy")  # set title
yticks(arange(0.5, 10, 1))  # set the ticks on the y-axis to line up with the energies of the states
for index, en in enumerate(energies):  # loop over the eigen-energies
    plot(grid, en + 5 * wavefunctions[:, index])  # plot each wavefunction, scaling and offsetting
```