Hand in at beginning of next lecture

In a simplified model of ammonia, the nitrogen atom can be in one of two places. The amplitude that it is on the left is $\psi_L$. The amplitude that it is on the right is $\psi_R$. We argued by symmetry that the equations of motion had to be of the form

$$i\partial_t \begin{pmatrix} \psi_L(t) \\ \psi_R(t) \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} \psi_L(t) \\ \psi_R(t) \end{pmatrix}. \quad (1)$$

We want to solve this differential equation with the initial conditions $\psi_L(0) = 1$ and $\psi_R(0) = 0$.

Problem 1.

1.1. The eigenvectors of the matrix

$$H = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad (2)$$

are

$$|+\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}. \quad (3)$$

What are the eigenvalues $E_+$ and $E_-$? [Note, the notation $|+\rangle$ is a common way to denote a column vector in an abstract state space. Physicist like to reserve the notations $\vec{v}$ or $\mathbf{v}$ for three dimensional vectors.]

Solution 1.1.
Problem 2. We can write
\[ |\psi(t)\rangle = \begin{pmatrix} \psi_L(t) \\ \psi_R(t) \end{pmatrix} = \alpha(t)|+\rangle + \beta(t)|-\rangle. \] (4)

2.1. Prove that if \( \alpha|+\rangle + \beta|\rangle = \gamma|+\rangle + \delta|\rangle \), then \( \alpha = \gamma \) and \( \beta = \delta \). This is known as the uniqueness of the eigenstate expansion.

Solution 2.1.

2.2. At time \( t = 0 \), the nitrogen atom is on the left: \( \psi_L(0) = 1 \) and \( \psi_R(0) = 0 \). Find \( a(0) \) and \( b(0) \).

Solution 2.2.
Problem 3.

3.1. Substitute Eq. (4) this into Eq. 1, and use the uniqueness of the eigenstate expansion, to find differential equations for $a(t)$ and $b(t)$.

Solution 3.1.

3.2. Solve these differential equations, and use the initial conditions to find $a(t)$ and $b(t)$.

Solution 3.2.
Problem 4.

4.1. What is the probability that at time $t$ the Nitrogen atom is on the right?

Solution 4.1.