Problem 1. Symmetric Bell’s Inequalities

In class we showed that quantum mechanics was incompatible with a local hidden variables theory. In particular, we imagined a setting where two spins were created in a completely correlated state. Each spin hits a detector. The detectors determine the polarization along one of three randomly chosen directions – each 120° from one another.

We showed that quantum mechanically, whenever the two detectors point in the same direction, the two spins will be detected to be aligned. [This is obvious – it is what we mean by completely correlated.] We argued that when the two detectors were pointing in different directions the spins will be “aligned” 25% of the time. By this I mean if one spin is “up” in the $\hat{z}$ direction, then the other will be up in the 120° basis 25% of the time. We will call this probability 25% the “coincidence probability”.

Finally we argued that this was inconsistent with a hidden variable theory.

In a hidden variable theory, each spin has an identical codebook, which tells it if it should be up or down in the three directions. Here you will calculate from symmetry the probabilities of different “codebooks.” In particular, let $P(\sigma_1, \sigma_2, \sigma_3)$ be the probability that the codebook has the spin being $\sigma_1$ in the $\hat{z}$ direction, $\sigma_2$ in the 120° direction, and $\sigma_3$ in the $-120°$ direction. [Note, the codebook is not probabilistic. This is the probability for a given codebook to be assigned to the spin.]

By symmetry

\[ P(\uparrow, \uparrow, \uparrow) = P(\downarrow, \downarrow, \downarrow) = p \]
\[ P(\uparrow, \uparrow, \downarrow) = P(\downarrow, \uparrow, \downarrow) = P(\uparrow, \downarrow, \uparrow) = P(\downarrow, \downarrow, \uparrow) = P(\uparrow, \downarrow, \downarrow) = P(\downarrow, \uparrow, \downarrow) = c. \]

1.1. Given that the sum of all probabilities is unity, write a relationship between $p$ and $c$.

1.2. Given that both $p$ and $c$ must be non-negative, what is the largest $c$ can be?

1.3. What is the smallest that the coincident probability can be:

\[ P_{\text{coinc}} = P(\uparrow, \uparrow, \uparrow) + P(\uparrow, \uparrow, \downarrow) + P(\downarrow, \downarrow, \uparrow) + P(\downarrow, \uparrow, \downarrow) \]

This is one form of “Bell’s Inequality” – and it is clearly violated by the quantum prediction (and the experiment).
Problem 2. Entanglement A state of two spins is called a product state if it can be written as
\[ |\psi\rangle = (a|\uparrow\rangle + b|\downarrow\rangle) \otimes (c|\uparrow\rangle + d|\downarrow\rangle). \] (4)
For a product state, a measurement of one spin tells you nothing about the state of the other.

For contrast, a state is called entangled if it cannot be written as a product state. An example is the Bell state
\[ |\psi\rangle = \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}. \] (5)

2.1. Prove that the Bell state is entangled. Hint: try to find what \(a, b, c, d\) are, and see if you get a contradiction.

2.2. Find one other entangled state, and prove that it is entangled.

Problem 3. Shor’s Algorithm

Here I will walk you through applying Shor’s algorithm for factoring the number 187. This is a small enough number that you can run the algorithm by hand – with minimal help from a classical computer.

3.1. The first step is to pick a random \(a\). Let’s choose \(a = 7\). We then define \(f(x) = \text{mod} (a^x, 187)\). Find the period of \(f\). [I recommend using a computer. The Python command for mod is \%.] Rather than just making a loop of powers of 7, then applying the mod operator to them, I would generate subsequent terms from the last via: \(f(x+1) = \text{mod}( a \cdot f(x), 187)\). By this I mean you have a loop where you at each time through the loop you take \(f \rightarrow \text{mod} (af, 187)\), and \(x \rightarrow x + 1\). You stop the loop as soon as \(f = 1\). The \(x\) which corresponds to that \(f\) is the period.

A good way to check that your loop is working is to try it with a simpler case, such as \(f(x) = \text{mod} (3^x, 7)\), which you can do by hand: \(f(0) = 1, f(1) = 3, f(2) = 2, f(3) = 6, f(4) = 4, f(5) = 5, f(6) = 1, f(7) = 3, \ldots\) This sequence has period 6 (which your program should be able to find).

If the loop stymies you, the other approach is to just generate a few hundred values of \(f\), and look for the first repeat.

The role of quantum mechanics in Shor’s algorithm is finding this period. The whole idea is that one can use quantum interference to find peaks in the discrete Fourier transform of this sequence, which lets you deduce the period.

3.2. If \(r\) is the period, and \(r\) is even, then Number Theory tells us that \(f(r/2)\) will have common factors with 187. Use this result to find the factors of 187. Here is a quick function which finds the greatest common divisor of two numbers:
def gcd(a,b):
    while b:
        a,b=b,a%b
    return a

This algorithm was known to the Greeks, and is called “Euclid’s Algorithm.”

Problem 4. Rutherford discovered the nucleus in 1911 by bombarding metal foils with α particles. Alpha particles are simply helium nuclei (made of 2 neutrons and 2 protons). An alpha particle has positive charge 2e. He expected the alpha particles to pass right through the foils with only small deviations. Most did, but some scattered off a very large angles. From the observations he concluded that most of the positive charge in the nucleus was confined to a small region. We will try to model the experiment and estimate the size of the nucleus.

4.1. Suppose the positive charge in the nucleus is spread over a distance \( a \). What should be (roughly) the maximum height of the potential seen by the \( \alpha \) particles (in terms of \( a \) and fundamental constants). I just want an order of magnitude – for example, feel free to argue on dimensional grounds. If you are stuck, you undoubtedly solved this problem in your intro mechanics class or your intro E&M class by using Gauss’s Law. Don’t spend too long on this – we just need an order of magnitude.

4.2. Sketch the potential felt by the \( \alpha \) particle.

4.3. Rutherford knew that his \( \alpha \) particles had a kinetic energy of 7.5MeV. Using classical reasoning, and the fact that some of the \( \alpha \) particles bounced back, put a bound on the maximum size of the nucleus (in meters). I just want an order of magnitude.

4.4. What would Rutherford do to put more stringent bounds on the size of the nucleus?

Problem 5.

5.1. What are the values of \( s \) such that
\[
e^{is} = 1?
\] (6)

Hint: There are infinitely many of them.

5.2. For given \( \theta \), what are the values of \( s \) such that
\[
e^{is} = e^{i\theta}.
\] (7)

Hint: Question 5.1 is the special case \( \theta = 0 \), so there should be infinitely many solutions.

5.3. For fixed real \( A \) and \( \phi \), what are the values of \( s \) and \( t \) such that
\[
e^{is+t} = Ae^{i\phi}?
\] (8)
One often summarizes these results by defining the natural log to be a multi-valued function,

\[
\ln(z) = \ln(|z|) + i \arg(z) + 2\pi i n. \tag{9}
\]

Your calculator typically gives this quantity with \( n = 0 \), called the “principal branch.” Similar, arcsin and arccos, defined as the inverse of sin and cos are most naturally thought of as multivalued functions, and your calculator only gives the principal branch.

**Problem 6.** Prove that

\[
\frac{\partial f}{\partial z} = 0 \tag{10}
\]

\[
\frac{\partial f}{\partial z^*} = 0. \tag{11}
\]

is equivalent to

\[
\frac{\partial f}{\partial x} = 0 \tag{12}
\]

\[
\frac{\partial f}{\partial y} = 0 \tag{13}
\]

for any function \( f \), where \( z = x + iy \) and \( z^* = x - iy \). Hint: Add and subtract the two equations.

**Problem 7. Feedback**

7.1. How long did this homework take?

7.2. Which of the following words come to mind when you think about this homework (feel free to add your own words if you have something better): frustrating, fun, tedious, insightful, hard, easy, useful, useless, fair, unfair