

## P3317 Final Exam

Due Dec 11 2018, 4:30pm

You may use your notes or other books. You may not get help from other people.

**Problem 1.** List the references you used to help solve these problems.

### Problem 2. Two particles in a double well

I was at a conference earlier this year where scientists from Germany demonstrated a new imaging system which allowed them to measure the position and spin of individual atoms. They illustrated the capabilities by studying two atoms in a double-well potential. The paper is: <https://arxiv.org/pdf/1807.06405.pdf>.

**2.1.** They were working with spin-1/2 fermions. Which of the following  $\psi_A, \psi_B, \psi_C, \psi_D$  are wavefunctions which could describe states of their system:

$$\psi_A(x_1, x_2, \sigma_1, \sigma_2) = \left[ e^{-\frac{(x_1-a)^2}{2d^2}} e^{-\frac{(x_2+a)^2}{2d^2}} - e^{-\frac{(x_2-a)^2}{2d^2}} e^{-\frac{(x_1+a)^2}{2d^2}} \right] (\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} - \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}) \quad (1)$$

$$\psi_B(x_1, x_2, \sigma_1, \sigma_2) = \left[ e^{-\frac{(x_1-a)^2}{2d^2}} e^{-\frac{(x_2+a)^2}{2d^2}} + e^{-\frac{(x_2-a)^2}{2d^2}} e^{-\frac{(x_1+a)^2}{2d^2}} \right] (\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} - \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}) \quad (2)$$

$$\psi_C(x_1, x_2, \sigma_1, \sigma_2) = \left[ e^{-\frac{(x_1-a)^2}{2d^2}} e^{-\frac{(x_2+a)^2}{2d^2}} - e^{-\frac{(x_2-a)^2}{2d^2}} e^{-\frac{(x_1+a)^2}{2d^2}} \right] (\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} + \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}) \quad (3)$$

$$\psi_D(x_1, x_2, \sigma_1, \sigma_2) = \left[ e^{-\frac{(x_1-a)^2}{2d^2}} e^{-\frac{(x_2+a)^2}{2d^2}} + e^{-\frac{(x_2-a)^2}{2d^2}} e^{-\frac{(x_1+a)^2}{2d^2}} \right] (\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} + \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}). \quad (4)$$

Here  $x_1$  and  $x_2$  are real numbers which represent space, while  $\sigma_1$  and  $\sigma_2$  take on the values  $\uparrow$  and  $\downarrow$ . The Kroeneker delta  $\delta_{\sigma\tau}$  is equal to 1 if  $\sigma = \tau$  and zero otherwise. The parameter  $2a$  represents the separation between the wells, and  $d \ll a$  represents the width of the wavefunction centered around each well.

If none of the wavefunctions can represent spin-1/2 fermions in a double well system, write a wavefunction that can.

**2.2.** Experiments of this form can also be done with bosons. Which of the wavefunctions,  $\psi_A, \psi_B, \psi_C, \psi_D$  could describe states of “spin-1/2” bosons in a double well? If none of the wavefunctions can represent spin-1/2 bosons in a double well system, write a wavefunction that can.

[Note: this is not a trick question. You may have learned at some point that bosons always have integer spin while fermions always have half-integer spin. Nonetheless, it is possible to have bosons whose low energy internal states act like spin-1/2.]

**2.3.** Do any of the wavefunctions represent states which are neither bosonic nor fermionic? If so, list them. Otherwise write a wavefunction for two spin-1/2 particles in a double well, which is neither fermionic nor bosonic.

**2.4.** Are any of the wavefunctions,  $\psi_A, \psi_B, \psi_C, \psi_D$ , eigenstates of the total spin projection operator  $\hat{S}_z = \hat{S}_{z1} + \hat{S}_{z2}$ ? If so, list them and their eigenvalues. If not, construct a two particle wavefunction which is an eigenstate of  $\hat{S}_z$ .

**2.5.** Are any of the wavefunctions,  $\psi_A, \psi_B, \psi_C, \psi_D$ , eigenstates of the operator  $\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 = \hat{S}_{x1}\hat{S}_{x2} + \hat{S}_{y1}\hat{S}_{y2} + \hat{S}_{z1}\hat{S}_{z2}$ ? If so, list them and the eigenvalues. If not, construct a wavefunction which is an eigenstate of  $\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$ .

**2.6.** Consider now the following wavefunctions,

$$\psi_E(x_1, x_2, \sigma_1, \sigma_2) = \left[ e^{-\frac{(x_1-a)^2}{2d^2}} e^{-\frac{(x_2-a)^2}{2d^2}} - e^{-\frac{(x_1+a)^2}{2d^2}} e^{-\frac{(x_2+a)^2}{2d^2}} \right] (\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} - \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}) \quad (11)$$

$$\psi_F(x_1, x_2, \sigma_1, \sigma_2) = \left[ e^{-\frac{(x_1-a)^2}{2d^2}} e^{-\frac{(x_2-a)^2}{2d^2}} + e^{-\frac{(x_1+a)^2}{2d^2}} e^{-\frac{(x_2+a)^2}{2d^2}} \right] (\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} - \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}) \quad (12)$$

$$\psi_G(x_1, x_2, \sigma_1, \sigma_2) = \left[ e^{-\frac{(x_1-a)^2}{2d^2}} e^{-\frac{(x_2-a)^2}{2d^2}} - e^{-\frac{(x_1+a)^2}{2d^2}} e^{-\frac{(x_2+a)^2}{2d^2}} \right] (\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} + \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}) \quad (13)$$

$$\psi_H(x_1, x_2, \sigma_1, \sigma_2) = \left[ e^{-\frac{(x_1-a)^2}{2d^2}} e^{-\frac{(x_2-a)^2}{2d^2}} + e^{-\frac{(x_1+a)^2}{2d^2}} e^{-\frac{(x_2+a)^2}{2d^2}} \right] (\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} + \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}). \quad (14)$$

Label each of these as bosonic, fermionic, or other.

**2.7.** The experimentalists repeatedly generate the same quantum state, and then measure the positions of each particle. In this manner they build up a histogram of the probability density. Take one of  $\psi_A, \psi_B, \psi_C, \psi_D$ , and use a computer to generate a density plot, where the axes are  $x_1$  and  $x_2$ , and color represents the probability density of finding the particles at positions  $x_1$  and  $x_2$ . Label your axes. As long as the color scheme is self-explanatory, there is no need to include a legend or key. State which of these wavefunctions you are using. Choose your scale, and the value of  $d/a$  appropriately. I found that choosing  $d/a = 0.3$  made a nice plot.

You should see multiple peaks. Explain the physical meaning of each peak.

**2.8.** Take one of  $\psi_E, \psi_F, \psi_G, \psi_H$ , and use a computer to generate a density plot, where the axes are  $x_1$  and  $x_2$ , and color represents the probability density of finding the particles at positions  $x_1$  and  $x_2$ . Label your axes. As long as the color scheme is self-explanatory, there is no need to include a legend or key. State which of these wavefunctions you are using.

You should see multiple peaks. Explain the physical meaning of each peak.

**2.9.** Suppose an experiment starts with the particles in state  $\psi_B$ . Which of the following best represents the probability that exactly one particle will be found near  $x = -a$ . [Take  $d \ll a$ , and by “near” I really mean “nearer to  $x = -a$  than  $x = a$ ”].

(A) 100%, (B) 75%, (C) 50%, (D) 25%, (E) 0%.

**2.10.** Suppose an experiment starts with the particles in state  $\psi_B$ . Suppose that the experimentalist measures near  $x = -a$  and finds a particle whose spin is  $\uparrow$ . Immediately after the measurement, what will the wavefunction  $\psi(x_1, x_2, \sigma_1, \sigma_2)$  be? Do not worry about normalization.

**2.11.** Now we want to understand the energy eigenstates of this system. To simplify our modeling, let's neglect spin and not worry about statistics. Thus, we want to study a two-particle wavefunction  $\psi(x_1, x_2)$ . Interactions between atoms are short ranged, so a good model for the inter-atomic interactions is a delta-function. Thus if we want to find energy eigenstates, we can write a time independent Schrodinger equation,

$$E\psi(x_1, x_2) = \left[ -\frac{\hbar^2}{2m}\partial_1^2 - \frac{\hbar^2}{2m}\partial_2^2 + V(x_1) + V(x_2) + g\delta(x_1 - x_2) \right] \psi(x_1, x_2). \quad (18)$$

where  $g$  parameterizes the interactions, and

$$V(x) = \frac{V_0}{a^4} (x^2 - a^2)^2 \quad (19)$$

represents the trapping potential, parameterized by  $V_0$  and  $a$ .

What are the dimensions of  $g, a, E, V_0, \hbar^2/m$ ?

**2.12.** By rescaling variables, one can write Eq. (18) as

$$\epsilon\psi(s_1, s_2) = \left[ -\frac{1}{2} [\partial_{s_1}^2 + \partial_{s_2}^2] + \alpha [(s_1^2 - 1)^2 + (s_2^2 - 1)^2] + \beta\delta(s_1 - s_2) \right] \psi(s_1, s_2). \quad (25)$$

Find how  $\epsilon, s_1, s_2, \alpha, \beta$  are related to  $x_1, x_2, g, a, E, V_0, \hbar^2/m$ .

**2.13.** We can use the variational principle to approximate the ground state solution to Eq. (25). Write down the functional which needs to be minimized.

**2.14.** Consider a variational wavefunction of the form

$$\psi(s_1, s_2) = A \left[ e^{-(s_1-1)^2/2d^2} e^{-(s_2+1)^2/2d^2} + e^{-(s_1+1)^2/2d^2} e^{-(s_2-1)^2/2d^2} \right] \quad (39)$$

$$+ B \left[ e^{-(s_1-1)^2/2d^2} e^{-(s_2-1)^2/2d^2} + e^{-(s_1+1)^2/2d^2} e^{-(s_2+1)^2/2d^2} \right]. \quad (40)$$

Explain in words what this state represents in the two different extreme cases  $A = 0$  and  $B = 0$ . Assume  $d \ll 1$ .

**2.15.** One expects the two extreme situations to correspond to very large positive  $\beta$  and very large negative  $\beta$ . If  $\beta$  is large and positive, do you expect  $A = 0$  or  $B = 0$ ?

Calculating the integrals you wrote down in question 2.13 is straightforward, if tedious. The subsequent minimization is also straightforward (and also tedious). I assure you that you have the tools to do this – and if you did you could verify your result from question 2.15. I would rather test other skills, so I will not ask you to do this.

If you read the experimental paper, you will see that part of the story they tell is how images like those you created in questions 2.7 and 2.8 change as one controls the interaction strength.

**Problem 3. Phonon Contribution to Specific Heat** One of the less flashy motivators for quantum mechanics was the specific heat of solids. The classical theory was unable to explain the experimental evidence.

In this problem you will calculate the dominant contribution to the specific heat of a solid at room temperature – which is due to acoustic phonons. Specific heat is a measure of how much energy it takes to raise the temperature:

$$C = \frac{\partial E}{\partial T}. \quad (41)$$

We will start by calculating the energy stored in acoustic phonons at temperature  $T$ .

**3.1.** Acoustic phonons have a dispersion  $\omega = c_s k$ , where  $c_s$  is the speed of sound. Following arguments that you used in your homework in discussing photons, the energy can be written as

$$E \propto \frac{\hbar}{c_s^3} \int_0^\infty d\omega \frac{\omega^\eta}{e^{\beta\hbar\omega} - 1}, \quad (42)$$

where  $\eta$  is a number. Show that the energy is of this form, and find  $\eta$ . We are not going to need the prefactor in front of the integral, so feel free to use proportional signs in your argument, and neglect multiplicative factors. I do want the whole argument though – it is insufficient to just say "EQ. xx of HW xx says ..." Be sure to list any resources you use in question 1.

**3.2.** By a change of variables, argue that the temperature dependence should be  $E \sim T^n$  for some  $n$ . Find  $n$  in terms of  $\eta$ .

**3.3.** Argue that the specific heat,  $C$ , should also be a power law in temperature. What is the power (in terms of  $\eta$ )?

**3.4.** It turns out that the specific heat from electrons in a metal is linear in  $T$ . At low  $T$ , which should dominate the specific heat: phonons or electrons? Explain.

**3.5.** It turns out that the linear dispersion of phonons is only good at low energies. Thus, a slightly more sophisticated model is

$$E \propto \frac{\hbar}{c_s^3} \int_0^{\omega_D} d\omega \frac{\omega^\eta}{e^{\beta\hbar\omega} - 1}. \quad (51)$$

The physics is that phonons cannot have arbitrarily large energy. The integral is therefore cut off by the "Debye" frequency  $\omega_D$ . For  $k_B T \ll \hbar\omega_D$  this cutoff can be ignored – and the expression in Eq. (42) is accurate. At high temperature  $\beta\omega_D \ll 1$ , we can expand the exponential  $e^{\beta\omega} \approx 1 + \beta\omega$ . In this high temperature limit, we again get a power law  $E \sim T^m$ . Find  $m$ .

**Problem 4. Cavity Quantum Electrodynamics** Recall that in classical mechanics, a harmonic oscillator of mass  $m$  and angular frequency  $\omega$  obeys the equations of motion  $dx/dt = p/m$ ,  $dp/dt = -m\omega^2 x$  where  $x$  is the position and  $p$  is the momentum. Using the dimensionless "reduced" variables  $X(t) = x(t)\sqrt{m\omega/\hbar}$  and  $P(t) = p(t)/\sqrt{m\hbar\omega}$ , the equations of motion are

$$\frac{dX}{dt} = \omega P \quad (54)$$

$$\frac{dP}{dt} = -\omega X, \quad (55)$$

and the energy is

$$U(t) = \frac{\hbar\omega}{2} (X^2(t) + P^2(t)) \quad (56)$$

Given all of our discussion in this course, it should seem intuitive that one can map a mode of the electromagnetic field onto such a harmonic oscillator. By analogy, a quantum mechanical description of that mode should be the same as the quantum mechanical description of a simple harmonic oscillator. In this problem you will go through this argument, then re-address some of our modeling of stimulated emission and stimulated absorption. You will find a new phenomenon: spontaneous emission, whereby an excited atom spontaneously de-excites, emitting a photon into a cavity.

In particular, consider a cavity for electromagnetic waves, of volume  $V$ . Throughout this problem we will consider a single mode of the electromagnetic field, of the form

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}, t) = \hat{\mathbf{x}}e(t) \sin(kz) \quad (57)$$

$$\vec{\mathbf{B}}(\vec{\mathbf{r}}, t) = \hat{\mathbf{y}}b(t) \cos(kz) \quad (58)$$

where  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$  are unit vectors in the x and y directions. The wave-number  $k$  is fixed.

Recall that Maxwell's equations in vacuum read

$$\nabla \cdot \mathbf{E}(\vec{\mathbf{r}}, t) = 0 \quad (59)$$

$$\nabla \times \mathbf{E}(\vec{\mathbf{r}}, t) = -\frac{\partial \mathbf{B}(\vec{\mathbf{r}}, t)}{\partial t} \quad (60)$$

$$\nabla \cdot \mathbf{B}(\vec{\mathbf{r}}, t) = 0 \quad (61)$$

$$\nabla \times \mathbf{B}(\vec{\mathbf{r}}, t) = \frac{1}{c^2} \frac{\partial \mathbf{E}(\vec{\mathbf{r}}, t)}{\partial t} \quad (62)$$

$$(63)$$

and that the total energy  $U(t)$  of the field in the cavity is

$$U(t) = \int_V d^3r \left( \frac{\epsilon_0}{2} E^2(\vec{\mathbf{r}}, t) + \frac{1}{2\mu_0} B^2(\vec{\mathbf{r}}, t) \right), \quad (64)$$

where the integral is taken over the cavity.

**4.1.** Use Maxwell's equations to produce expressions for  $de/dt$  and  $db/dt$  as functions of  $k, c, e(t)$ , and  $b(t)$ ,

**4.2.** Express  $U(t)$  in terms of  $V, e(t), b(t), \epsilon_0$ , and  $\mu_0$ . Use that

$$\int_V d^3r \sin^2 kz = \int_V d^3r \cos^2 kz = \frac{V}{2} \quad (69)$$

**4.3.** Setting  $\omega = ck$ , and introducing the reduced variables

$$\chi(t) = \sqrt{\frac{\epsilon_0 V}{2\hbar\omega}} e(t) \quad (73)$$

$$\Pi(t) = \sqrt{\frac{V}{2\mu_0\hbar\omega}} b(t), \quad (74)$$

show that the equations for  $d\chi/dt$ ,  $d\Pi/dt$  and  $U(t)$  map onto Eqs. (54)-(56). Use that  $\epsilon_0\mu_0 = 1/c^2$

**4.4.** Given that the equations of motion are the same for the electromagnetic mode, and the simple harmonic oscillator, it is reasonable to posit that they have the same quantum description. In particular, one can write down a wave-function  $\psi(\chi)$ , where  $\chi$ , the dimensionless electric field plays the role of the dimensionless position  $X$ . We can define the operator  $\hat{\chi}$ , which has the property  $\hat{\chi}\psi(\chi) = \chi\psi(\chi)$ . Similarly the scaled magnetic field plays the role of the momentum, and in this representation can be expressed as  $\hat{\Pi}\psi(\chi) = -i\partial_\chi\psi(\chi)$ .

Given this definition of the operators, calculate the commutator  $[\hat{\chi}, \hat{\Pi}]$ .

**4.5.** In terms of these operators, the energy (and hence the Hamiltonian operator) is

$$\hat{H}_C = \frac{\hbar\omega}{2} (\hat{\chi}^2 + \hat{\Pi}^2). \quad (92)$$

Write  $\hat{H}_C$  in terms of the operators

$$\hat{a} = \frac{\hat{\chi} + i\hat{\Pi}}{\sqrt{2}} \quad (93)$$

$$\hat{a}^\dagger = \frac{\hat{\chi} - i\hat{\Pi}}{\sqrt{2}} \quad (94)$$

**4.6.** Following the arguments that you are familiar with from the simple Harmonic oscillator, the eigenstates of this Hamiltonian have energy  $\hbar\omega(n + 1/2)$ , and can be written as  $|n\rangle$ , where

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad (102)$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle. \quad (103)$$

The index  $n$  is interpreted as the “number of photons” in the cavity, and the operator  $\hat{N} = \hat{a}^\dagger\hat{a}$  measures this quantity. The electric and magnetic fields can be re-expressed as

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \hat{\mathbf{x}}\sqrt{\frac{\hbar\omega}{\epsilon_0 V}}(\hat{a} + \hat{a}^\dagger)\sin kz \quad (104)$$

$$\vec{\mathbf{B}}(\vec{\mathbf{r}}) = i\hat{\mathbf{y}}\sqrt{\frac{\mu_0\hbar\omega}{V}}(\hat{a} - \hat{a}^\dagger)\cos kz. \quad (105)$$

Calculate  $\langle n|\vec{\mathbf{E}}(\vec{\mathbf{r}})|n\rangle$  and  $\langle n|\vec{\mathbf{E}}(\vec{\mathbf{r}}) \cdot \vec{\mathbf{E}}(\vec{\mathbf{r}})|n\rangle$ . This result is equivalent to calculating the expectation value of the position, and the position squared, in a harmonic oscillator energy eigenstate.

**4.7.** Now consider an atom at point  $\vec{\mathbf{r}}_0$  in the cavity. We will neglect the motion of the atom, or consider it classically. Rather we are concerned with two of its internal states: the ground state  $|f\rangle$ , and an excited state  $|e\rangle$ . The origin of energy is chosen so that these states have energy

$$\hat{H}_A|f\rangle = -\frac{\hbar\omega_a}{2}|f\rangle \quad (111)$$

$$\hat{H}_A|e\rangle = \frac{\hbar\omega_a}{2}|e\rangle. \quad (112)$$

It is natural to introduce Pauli operators,

$$\hat{\sigma}_z|e\rangle = |e\rangle \quad \hat{\sigma}_z|f\rangle = -|f\rangle \quad (113)$$

$$\hat{\sigma}_+|e\rangle = 0 \quad \hat{\sigma}_+|f\rangle = |e\rangle \quad (114)$$

$$\hat{\sigma}_-|e\rangle = |f\rangle \quad \hat{\sigma}_-|f\rangle = 0, \quad (115)$$

in which case  $\hat{H}_A = (\hbar\omega_A/2)\hat{\sigma}_z$ .

A basis for the combined states of the cavity and atom can be written as  $|f, n\rangle, |e, n\rangle$ , with  $n = 0, 1, \dots$ . Suppose the Hamiltonian is

$$\hat{H}_0 = \hat{H}_A + \hat{H}_C. \quad (116)$$

What is  $\hat{H}_0$  acting on each of the states  $|f, n\rangle$ , and  $|e, n\rangle$ .

**4.8.** For the remainder of the problem we will assume that the cavity is tuned so that  $\omega_a = \omega$ . Find the energies of the five lowest energy eigenstates of  $\hat{H}_0$ . Are there any degeneracies?

**4.9.** So far we have neglected the coupling between the electromagnetic field and the atom. A good model for this interaction is

$$\hat{W} = \gamma(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-) \quad (119)$$

where  $\gamma = -d\sqrt{\hbar\omega/\epsilon_0V}\sin kz_0$ , where  $d$  is the electric dipole matrix element between the two atomic states. For simplicity, we will assume that the atomic position is such that  $\sin kz_0 = 1$

Show that

$$\hat{W}|e, n\rangle = \lambda|f, n+1\rangle \quad (120)$$

$$\hat{W}|f, n+1\rangle = \lambda|e, n\rangle, \quad (121)$$

and find the coefficient  $\lambda$ .

**4.10.** What does  $\hat{W}$  do to the state  $|f, 0\rangle$ ?

**4.11.** Which of the processes  $\hat{a}\hat{\sigma}_+$  and  $\hat{a}^\dagger\hat{\sigma}_-$  represent the atom absorbing a photon from the cavity, and which represents emission?

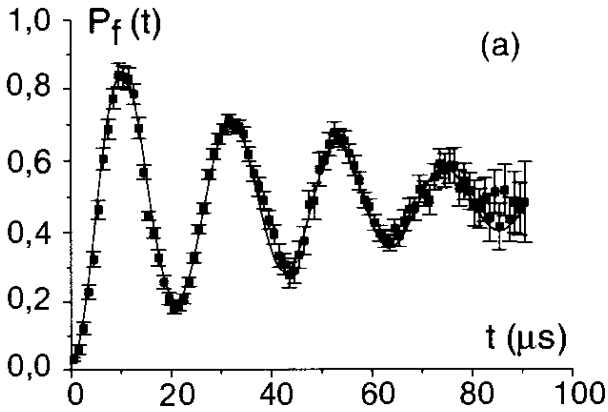
**4.12.** Determine the energy eigenstates of  $\hat{H} = \hat{H}_0 + \hat{W}$ . Hint: it requires constructing a sequence of  $2 \times 2$  matrices, and diagonalizing them. The diagonalization should be trivial.

**4.13.** Suppose an excited atom is introduced to an empty cavity at time  $t = 0$ . That is  $|\psi(t = 0)\rangle = |e, 0\rangle$ . Find the state of the system at time  $t$ .

**4.14.** What is the probability  $P_f(T)$  of finding the atom in state  $f$  at time  $T$ , when it exits the cavity?

**4.15.** The experiment has been performed on rubidium atoms with states  $(f, e)$  such that  $d = 1.1 \times 10^{-26}\text{Cm}$  and  $\omega/2\pi = 5 \times 10^{10}\text{Hz}$ . The cavity has volume  $1.87 \times 10^{-6}\text{m}^3$ . What is the smallest non-zero time (in ms) where  $P_f(t)$  vanishes? [As stated before, assume  $z_0$  is chosen so that  $\sin kz_0 = 1$ ].

**4.16.** The following graph shows the experimental results:



How do these experimental results compare with the theoretical prediction? What agrees and what disagrees? Include both qualitative and quantitative comparisons.

**4.17.** In this course we spent a lot of time thinking about modeling. The natural next step is to figure out what is missing in our model. While coming up with the correct explanation is often hard, it should be pretty easy to at least come up with something we missed.

Find at least one flaw in our model, or at least one piece of physics which we may have neglected.