P3317 HW from Lecture 10+11 and Recitation 6

Due Thursday October 11 – Note the delay due to Fall Break

Problem 1. Quantum Cryptography

[This was adapted from a 2014 P3317 final exam problem. It is a little lengthy – but is really cool.]

Cryptography involves sending a message to a correspondent while minimizing the risk of interception by an unwanted outsider. Measuring a quantum system disturbs it – a feature which can be used as part of a cryptography scheme.

Following convention we will label the person sending the message as "Alice," and use the symbol A to label quantities associated with her. The receiver is "Bob." A spy trying to listen in on the conversation is "Cathy." The message consists of a sequence of $+1$ and -1's. For example, letting $+$ denote a $+1$ and $-$ denote a -1, a message could be $++---++$.

Consider a spin 1/2 particle. The spin operator is $\mathbf{S} = (\hbar/2)\vec{\sigma}$, where the set $\sigma_i, i = x, y, z$ are Pauli matrices. We write $|\sigma_z = +1\rangle$ and $|\sigma_z = -1\rangle$ for the eigenstates of S_z with eigenvalues $\hbar/2$ and $-\hbar/2$.

Consider a particle in the state $|\sigma_z = +1\rangle$. As in our analysis of Bells inequalities, one can measure the component of the spin along an axis u in the (x, z) plane, defined by the unit vector

$$\hat{\mathbf{u}} = \cos(\theta)\hat{\mathbf{z}} + \sin(\theta)\hat{\mathbf{x}}. \tag{1}$$

The corresponding operator is

$$S_u = \mathbf{S} \cdot \hat{\mathbf{u}} = \frac{\hbar}{2}\left(\cos(\theta)\sigma_z + \sin(\theta)\sigma_x\right). \tag{2}$$

The possible results of the measurement are $+\hbar/2$ and $-\hbar/2$.

One sees this by noting that the relevant operator is

$$S_u = \frac{\hbar}{2}\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}. \tag{3}$$

We let λ_1 and λ_2 be the eigenvalues. The trace of this matrix is $\lambda_1 + \lambda_2 = 0$ and the determinant is $\lambda_1\lambda_2 = -\hbar^2/4$. Thus the eigenalues are $\lambda_2 = -\lambda_1 = \hbar/2$.

1.1. The eigenstates of the observable in Eq. (2) can be written

$$\begin{aligned} |\sigma_u = +1\rangle &= \cos(\phi)|\sigma_z = +1\rangle + \sin(\phi)|\sigma_z = -1\rangle & (4) \\ |\sigma_u = -1\rangle &= -\sin(\phi)|\sigma_z = +1\rangle + \cos(\phi)|\sigma_z = -1\rangle. & (5) \end{aligned}$$

Express ϕ in terms of θ.

1.2. In terms of ϕ, what is the probability p_+ that the measurement yields $+\hbar/2$?

1.3. If the measurement yields $+\hbar/2$, what will the spin state be immediately after the measurement?

1.4. In terms of ϕ, what is the probability p_- that the measurement yields $-\hbar/2$?

1.5. If the measurement yields $-\hbar/2$, what will the spin state be immediately after the measurement?

1.6. Suppose after measuring the observable in Eq. (2) and finding $+\hbar/2$ one measures S_z. What is the probability q_+ that one finds $\hbar/2$ for this second measurement?

1.7. Suppose after measuring the observable in Eq. (2) and finding $+\hbar/2$ one measures S_z. What is the probability q_- that one finds $-\hbar/2$ for this second measurement?

1.8. Imagine a sequence where one begins with the spin in the $|\sigma_z = +1\rangle$, measures S_u, then measures S_z. Show that the total probability of finding $+\hbar/2$ for this second measurement is

$$P_{++}(\theta) = p_+q_+ + p_-q_- = \frac{1+\cos^2(\theta)}{2}. \tag{15}$$

1.9. Assuming now that the initial state is $|\sigma_z = -1\rangle$. One measures S_u, then measures S_z. What is the total probability, $P_{--}(\theta)$ of finding $-\hbar/2$ for this second measurement?

1.10. Imagine now that we have two spins in the state

$$|\Sigma\rangle = \frac{1}{\sqrt{2}}\Big[|\sigma_z^a = +1\rangle \otimes |\sigma_z^b = +1\rangle + |\sigma_z^a = -1\rangle \otimes |\sigma_z^b = -1\rangle\Big]. \tag{22}$$

We saw this state when discussing Bell's inequalities. Show that this can also be written as

$$|\Sigma\rangle = \frac{1}{\sqrt{2}}\Big[|\sigma_x^a = +1\rangle \otimes |\sigma_x^b = +1\rangle + |\sigma_x^a = -1\rangle \otimes |\sigma_x^b = -1\rangle\Big]. \tag{23}$$

1.11. The pair of particles (a, b) is prepared in the spin state $|\Sigma\rangle$ in Eq. (22). As the two particles move away from one another, this spin state remains unchanged unless a measurement is made. Alice measures the spin component of a along an axis u_a defined by the angle θ_a. What are the possible results and the corresponding probabilities in the two cases $\theta_a = 0$ and $\theta_a = \pi/2$. These respectively correspond to the $\hat{z}$ and $\hat{x}$ axis.

1.12. After Alice's measurement, the spin state of the two particles will depend on the result of the measurement. For example, if Alice measures along $\hat{z}$ and finds $\hbar/2$ the spins will be in the state

$$|\uparrow\uparrow\rangle = |\sigma_z^a = +1\rangle \otimes |\sigma_z^b = +1\rangle.$$

If Alice measures along $\hat{z}$ and finds $-\hbar/2$ what will the spin state be?

1.13. If Alice measures along $\hat{x}$ and finds $\hbar/2$ the spins will be in the state

$$|\rightarrow\rightarrow\rangle = |\sigma_x^a = +1\rangle \otimes |\sigma_x^b = +1\rangle$$

. If Alice measures along $\hat{x}$ and finds $-\hbar/2$, what will the spin state be?

1.14. After Alice's measurement, Bob measures the spin of particle b along an axis u_b defined by the angle θ_b. Give the possible results of Bob's measurement and their probabilities in terms of Alice's results in the following four configurations:

1. $\theta_a = 0, \theta_b = 0$
2. $\theta_a = 0, \theta_b = \pi/2$
3. $\theta_a = \pi/2, \theta_b = 0$
4. $\theta_a = \pi/2, \theta_b = \pi/2$

1.15. In which of the four cases are the measurements on a and b guaranteed to give the same result?

1.16. A spy, Cathy, sits between the source and Bob. She measures the spin of particle b along an axis u_c defined by angle θ_c, then lets the spin continue on to Bob. Suppose $\theta_a = 0$. In terms of θ_c, and Alice's findings, what are the possible results of Cathy's measurements, and what are their possibilities? [You may find it useful to take advantage of your results from earlier questions.]

1.17. After Cathy's measurement, Bob measures the spin of b along the axis $\theta_b = 0$. In terms of Cathy's results, what does Bob find, and with what probability? [You may find it useful to take advantage of your earlier results.]

1.18. Suppose that $\theta_a = \theta_b = 0$. As before, Cathy measures the spin along the direction θ_c. What is the probability $P(\theta_c)$ that Alice and Bob find the same result after Cathy's measurement?

1.19. What is the expectation value of $P(\theta_c)$ if Cathy chooses θ_c at random in the interval $[0, 2\pi]$ with uniform probability?

1.20. What is the expectation value of $P(\theta_c)$ if Cathy chooses θ_c to be either 0 or $\pi/2$, each with probability 1/2.

We can now explain the quantum cryptography procedure

1. Alice prepares her n bit message
2. Alice, who controls the source, prepares an ordered sequence of $N \gg n$ pairs in the spin state $|\Sigma\rangle$. She sends all the b spins to Bob and keeps the a spins.
3. For each spin they collect, Alice and Bob measure either the x or the z component. For each spin they independently choose the x or z direction at random with probability $p = 1/2$. Alice's choice and Bob's choice are independent. The both register all their results.
4. Bob selects a subset of his measurements (say $N/2$ of them). He calls Alice on the telephone, and gives this list to Alice. He includes the axis, and the result of the measurement
5. Alice compares, for these $N/2$ spins, her axes and her results. By doing so she can tell if a spy is present. If a spy is spotted they abort the transmission.
6. For the other $N/2$ spins, Bob tells Alice which axes he measured, but he does not tell her his results.
7. Alice calls Bob on the phone, and ...

1.21. How can Alice be sure that a spy is present?

1.22. What is the probability that an operating spy will escape being detected? Calculate this probability for $N/2 = 200$. Make some reasonable assumptions. For example, assume Cathy randomly chooses θ_c independently for each spin, and use the results of 1.19 or 1.20. Explicitly state all your assumptions.

1.23. After one attempt to communicate, Alice measured $1 : x+, 2 : z+, 3 : z-, 4 : z+, 5 : x+, 6 : x-, 7 : z+, 8 : x+, 9 : x-, 10 : z-, 11 : x+, 12 : z+$ the number represents which spin, the letter represents the axis, and the $\pm$ gives the result of the measurement. Bob reported to Alice half of his results $2 : x+, 5 : x+, 8 : x-, 9 : z+, 11 : z+, 12 : z-$. Alice declared that there was a spy. How did she know?

1.24. In as separate attempt to communicate, Alice measured $1 : x+, 2 : x-, 3 : z+, 4 : x+, 5 : z-, 6 : z-, 7 : x+, 8 : z+, 9 : z+, 10 : z-, 11 : x+, 12 : x-$. Bob reports $1 : x+, 3 : x-, 4 : z-, 7 : x+, 10 : x+, 11 : x+$. Alice decides that it was unlikely that a spy was listening in. What is the probability that a spy was listening?

1.25. In this second case [where Alice measured $1 : x+, 2 : x-, 3 : z+, 4 : x+, 5 : z-, 6 : z-, 7 : x+, 8 : z+, 9 : z+, 10 : z-, 11 : x+, 12 : x-$ and Bob reported $1 : x+, 3 : x-, 4 : z-, 7 : x+, 10 : x+, 11 : x+$], Bob reports the following axes: $2 : x, 5 : x, 6 : x, 8 : z, 9 : x, 12 : x$. Alice wants to send the message $(+, -)$. What can she tell Bob on the phone to securely transmit this message? What will Bob do?

Problem 2. We wish to use the variational principle to estimate the ground state energy of the Hydrogen atom.

2.1. Using the techniques we have previously studied, adimensionalize the time independent Schrodinger equation for the electron in a Hydrogen atom

$$E\psi(r) = -\frac{\hbar^2}{2m}\nabla^2\psi(r) - \frac{e^2}{4\pi\epsilon_0}\frac{1}{r}\psi(r). \tag{38}$$

Call your scaled coordinate s.

2.2. Given an arbitrary normalized function of the scaled coordinate, $\psi(s)$, we can use the variational principle to produce an upper bound to the ground state energy of Hydrogen. By normalized, I mean

$$\int d^3s\, |\psi(s)|^2 = 1. \tag{39}$$

Write an expression for this variational bound. The expression will have the form

$$\frac{E}{E_0} = A\int d^3s\, |\nabla\psi(s)|^2 + B\int d^3s\, \frac{|\psi(s)|^2}{s}, \tag{40}$$

where E_0 is the energy scale used in adimensionalizing the Schrodinger equation, A, B are dimensionless constants. Find A and B.

2.3. We will try the following variational wavefunction

$$\psi(s) = \frac{1}{\sqrt{8\pi\lambda^3}} e^{-s/(2\lambda)}, \tag{41}$$

where λ is an undetermined variational parameter. This is is properly normalized as

$$\int d^3s\, |\psi(s)|^2 = 4\pi \int ds\, s^2 |\psi(s)|^2 = 1. \tag{42}$$

Calculate the variational energy $E(\lambda)$ in Eq. (40). Just use the symbols, A, B, and E_0. Do not substitute your expressions for these parameters.

2.4. Minimize $E(\lambda)$ with respect to λ. Again, just use the symbols A, B, and E_0.

2.5. Substitute in your values for A, B, and E_0. What is your bound on the energy in eV.

Problem 3. Consider the adimensionalized time independent Schrodinger equation for a a particle in a double well potential

$$E\psi(x) = -\frac{1}{2}\partial_x^2 \psi(x) + V_0(x^2 - 1)^2 \psi(x). \tag{43}$$

Take $V_0 = 20$.

3.1. Plot the potential. Label your graph.

3.2. Choose a reasonable spatial and temporal discretization (I took $dx = 0.05$ with x running from -5 to 5 and $dt = 0.01$, and it seemed to work for me). Make a stationary wave packet of width 0.3, centered at $x = -1$. Numerically integrate the time dependent Schrodinger equation until time $t = 600$. Make a properly labeled density plot where the horizontal axis is position, the vertical axis is time, and the brightness corresponds to $|\psi|^2$.

3.3. Describe in words what you observe. Is this result consistent with the modeling of Ammonia we used in terms of two-level systems?

Problem 4. Feedback

4.1. How long did this homework take?

4.2. Which of the following words come to mind when you think about this homework (feel free to add your own words if you have something better): frustrating, fun, tedious, insightful, hard, easy, useful, useless, fair, unfair