P3317 HW from Lecture 20+21 and Recitation 11

Due Nov 13, 2018

Problem 1. Parameters in our model of sound
As you know from the in-class exercise in lecture, this exercise is somewhat tedious, and requires careful book-keeping.

In class we modeled optical phonons via a mass-and-spring model

$$H = \sum_j \frac{p_j^2}{2m} + \frac{\kappa}{2} x_j^2 + \frac{\gamma}{2}(x_j - x_{j-1})^2. \tag{1}$$

We defined harmonic oscillator creation and annihilation operators,

$$\begin{aligned} x_j &= \frac{d}{\sqrt{2}} \left([(a_j + a_j^\dagger) + \alpha(a_{j+1} + a_{j+1}^\dagger + a_{j-1} + a_{j-1}^\dagger)\right] & (2) \\ p_j &= \frac{\hbar}{\sqrt{2} di} \left([(a_j - a_j^\dagger) - \alpha(a_{j+1} - a_{j+1}^\dagger + a_{j-1} - a_{j-1}^\dagger)\right]. & (3) \end{aligned}$$

In the limit $\gamma \ll k$ (and $\alpha \ll 1$) we arrived at

$$\begin{aligned} H = \sum_j \Big[& A(a_j^\dagger a_j + a_j a_j^\dagger) + B(a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}) \\ & + C(a_j a_j + a_j^\dagger a_j^\dagger) + D(a_j a_{j+1} + a_{j+1}^\dagger a_j^\dagger) + \cdots \Big]. \end{aligned} \tag{4}$$

I claimed that we can choose α and d to make $C = D = 0$.

1.1. Choose *one* of the following three energies

$$\begin{aligned} K &= \sum_j \frac{p_j^2}{2m} & (5) \\ V_\kappa &= \sum_j \frac{\kappa}{2} x_j^2 & (6) \\ V_\gamma &= \sum_j \frac{\gamma}{2}(x_j - x_{j-1})^2. & (7) \end{aligned}$$

Calculate the contribution to A, B, C, D from that one energies, and verify that there are no other terms necessary.

If you choose K or V_κ, include terms which are up to linear order in α, but neglect terms which are quadratic or higher. If you choose V_γ, neglect all terms which are linear or higher order in α.

Problem 2. Classical Sound Waves

Consider the model we studied in class:

$$H = \sum_j \frac{p_j^2}{2m} + \frac{\kappa}{2}x_j^2 + \frac{\gamma}{2}(x_j - x_{j-1})^2. \tag{8}$$

Lets take an infinite chain where n extends from $-\infty$ to $+\infty$. The classical equations of motion for this model are

$$\begin{aligned} m\partial_t x_j &= p_j && (9) \\ \partial_t p_j &= -\kappa x_j - \gamma(2x_j - x_{j+1} - x_{j-1}). && (10) \end{aligned}$$

In this problem you will solve these classical equations of motion, and find the frequencies ω of the classical modes. By the correspondence principle you expect the energy of of the quanta to be given by $\hbar\omega$.

To solve this classical problem, we guess a solution of the form $x_j = ue^{-i\omega t+ikaj}$, and $p_j = ve^{-i\omega t+ikaj}$ where u, v, ω, and j are constants, and the lattice spacing is a.

2.1. Substitute this ansatz into equation (9) and (10). You will have two equations with three unknowns u, v, ω. Solve one of the equations for u, and substitute it into the other. You should get an equation which uniquely defines ω.

[Alternatively, you can recognize your two equations as an eigenvalue problem for ω. The above procedure is just a trick for solving that eigenvalue problem.]

2.2. Taylor expand your expression for small k, to get an expression of the form $\omega^2 = A + Bk^2 + \cdots$. Given that classical frequencies are related to quantum mechanical energies, this is reminiscent of the relativistic formula $E^2 = p^2c^2 + M^4c^4$. By equating $E = \hbar\omega$, and $p = \hbar k$, find the "Mass gap" M and the "effective speed of light c. Because of this analogy, these optical modes are sometimes called "Massive". [For contrast, the acoustic modes have no gap, and are called "massless."]

Problem 3. Baryons

Baryons are bound states of three quarks. They labeled by their quark composition, and their total angular momentum J. The"light" baryons are made of u, d, or s.

3.1. Look up the names of the 10 light baryons which have $J = 3/2$. List each of their quark composition.

3.2. Look up the names of the 8 light baryons which have $J = 1/2$. List each of their quark composition. (Note, both the neutral lambda and the neutral sigma have the same quark composition)

3.3. In a sentence or two, argue for why are there more light baryons with $J = 3/2$ than with $J = 1/2$? [Hint: think about the case uuu, and what the spin wavefunction looks like.]

Problem 4. Flavor Neutral Mesons
In this question we will consider the flavor neutral mesons.

Mesons are bound states of a quark and anti-quark. A meson is "flavor neutral" if it has no net flavor – for example the bound state of and up and anti-up quark, or the bound state of a down and anti-down. Since these mesons all have the same "flavor" the strong interactions can mix them. That is, the energy eigenstate is a superposition of the various combinations.

The simplest model of these mesons is in terms of a three component wavefunction

$$|\psi\rangle = \psi_u|u\bar{u}\rangle + \psi_d|d\bar{d}\rangle + \psi_s|s\bar{s}\rangle, \tag{11}$$

where $|\psi_u|^2, |\psi_d|^2, |\psi_s|^2$ is the probability that the meson is in the $u\bar{u}, d\bar{d}$, or $s\bar{s}$ state.

The mass (energy) of the π^0, η and η' will be the eigenvalues of a 3-component Schrodinger equation

$$H\begin{pmatrix}\psi_u\\ \psi_d\\ \psi_s\end{pmatrix} = E\begin{pmatrix}\psi_u\\ \psi_d\\ \psi_s\end{pmatrix}, \tag{12}$$

where H is a 3×3 matrix, and E is identified with M. The eigenvectors give the flavor decomposition of these three mesons. Experimentally these mesons have mass 135MeV, 548MeV, and 958MeV.

4.1. A first stab at modeling H is to assume that the standard model treats u, d, s quarks all the same. We know that this is not completely true, but one can hypothesize that it is an approximate symmetry. One calls this (hypothetical) symmetry $SU(3)$ flavor symmetry – this sounds fancy, but it just means that the theory should be invariant under permuting the flavor among these 3 possibilities. The most general H for this meson problem which treats u, d, and s the same is

$$H_{SU(3)} = \begin{pmatrix} a & b & b\\ b & a & b\\ b & b & a\end{pmatrix}. \tag{13}$$

Find the eigenvalues of this matrix.

You should find that two of them are the same. This tells you that if $SU(3)$ flavor symmetry was exact, the mass of the π^0 should be equal to the mass of the η.

4.2. The fact that the π_0 and η have different energies tells you that $SU(3)$ flavor symmetry is broken. One hypothesis is that you can account for the symmetry breaking solely by giving the s quarks a higher mass. This would be modeled by a Hamiltonian

$$H_{qqs} = \begin{pmatrix} a & b & b\\ b & a & b\\ b & b & c\end{pmatrix}. \tag{14}$$

One could then fit a, b, c to try to match the energies of the π_0, η and η' as closely as possible. I tried this procedure and came up with $a = 340$MeV, $b = 200$MeV, and $c = 938$MeV. Using these values for a, b, c, find the mass of the light flavor neutral mesons. [Feel free to use a computer algebra system.]

You should find this result is a lot better than the $SU(3)$ symmetric theory, but is still not perfect. Apparently the symmetry breaking can not all be accounted for by the mass of the strange quark. Given that we only have 3 pieces of data here (the masses), adding more parameters to our model is probably not wise.

4.3. Within the model in Eq. (14) with the parameters $a = 340$MeV, $b = 200$MeV, and $c = 938$MeV, what are the flavor wavefunctions for the π_0, η, η'. [Again, you may want to use a conputer.] In what way do these differ from the nominal flavor wavefunctions listed in the meson tables?

4.4. Suppose you created a meson in the $u\bar{u}$ state. Estimate what the characteristic time (in seconds) would be for it to oscillate into $d\bar{d}$? If the meson is moving at the speed of light, how far would it move (in meters) during that time? [You may find it useful to look back on the homework problem about the K mesons. You don't need to go through the whole argument here, just estimate the timescale.]

Problem 5. Neutrino background

In the early universe, the matter and photons were in thermal equilibrium. As the temperature fell to below $T \sim 10^{11}$K the neutrinos decoupled from the other particles. In this regime the neutrinos can be treated as noninteracting, massless, relativistic particles. Thus their energy density should be roughly the same as black-body radiation. [It will be a bit different because they are Fermions, but using the results for photons will not be too far off.]

5.1. Estimate the energy density in neutrinos at the time of decoupling. [It might help to look at your homework on the Planck Spectrum. Feel free to use results you derived there – you don't need to rederive the Plank Spectrum.] Quote your answer in eV/m^3. You can use the fact that

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}. \tag{15}$$

Include the fact that there are 3 flavors of neutrino, and 3 of antineutrino (but only one spin state of each).

5.2. The mass of a proton is roughly 1GeV. What density of Hydrogen atoms (in atoms per cubic meter) would you need so that the energy density from their rest mass is equal your result in 5.1. This will give you a feel for how big your answer is.

5.3. We actually know neutrinos have mass. Suppose the neutrino mass is about $m_\nu \sim 10^{-3}$eV. Assuming you have a gas of neutrinos at temperature T. At what temperature (in Kelvin) would you need to take into account the mass?

Problem 6. Feedback

6.1. How long did this homework take?

6.2. Which of the following words come to mind when you think about this homework (feel free to add your own words if you have something better): frustrating, fun, tedious, insightful, hard, easy, useful, useless, fair, unfair