P3317 HW from Lecture 24+25 and Recitation 13

Due Nov 27, 2018

Problem 1. The Higgs Mechanism in 1+1 dimensional Quantum Electrodynamics
In lecture we explicitly went through a calculation of the Higgs mechanism for 2+1 dimensional scalar quantum electrodynamics. Recall the Higgs mechanism describes the physics of a gauge field coupled to a "Higgs" field. When the Higgs field acquires a finite mean value, the gauge modes become gapped. This is the model used to explain the masses of the W and Z bosons in the theory of the weak interactions. Here I will have you go through the same argument in 1+1 dimension.

We will use x to label the spatial dimension, and t the time dimension. We have one scalar field $\phi(x,t)$, the Higgs field. We will have two gauge fields: A_t and A_x. In one spatial dimension there is no analog of a magnetic field, but you can have an electric field. It is defined by

$$E = \partial_t A_x - \partial_x A_t. \tag{1}$$

In lecture we used the symbol S_y for this quantity.

We will start with the simplest case where the Higgs field obeys a relativistic wave equation

$$(\partial_t - iA_t)^2\phi - (\partial_x - iA_x)^2\phi + m^2\phi = 0. \tag{2}$$

The gauge fields obey "Maxwell" equations

$$\begin{aligned} \partial_x E + \frac{1}{2}\left[i\phi^*(\partial_t - iA_t)\phi - i\phi(\partial_t + iA_t)\phi^*\right] &= 0 \qquad (3)\\ -\partial_t E - \frac{1}{2}\left[i\phi^*(\partial_x - iA_x)\phi - i\phi(\partial_x + iA_x)\phi^*\right] &= 0 \end{aligned}$$

1.1. Show that these three equations [Eqs. (2)-(3)] possess a Gauge symmetry. That is they are invariant under the transformation

$$A_x \rightarrow A_x + \partial_x\Lambda \tag{4}$$
$$A_t \rightarrow A_t + \partial_t\Lambda \tag{5}$$
$$\phi \rightarrow e^{i\Lambda}\phi. \tag{6}$$

Hint: First show that

$$(\partial_t - i(A_t + \partial_t\Lambda))e^{i\Lambda}\phi = e^{i\Lambda}(\partial_t - iA_t)\phi.$$

Next define $\psi = (\partial_t - iA_t)\phi$, and reuse your first result:

$$(\partial_t - i(A_t + \partial_t\Lambda))e^{i\Lambda}\psi = e^{i\Lambda}(\partial_t - iA_t)\psi.$$

Finally, putting these together yields

$$(\partial_t - i(A_t + \partial_t\Lambda))^2 e^{i\Lambda}\phi = e^{i\Lambda}(\partial_t - iA_t)^2\phi.$$

With these identities (and the similar ones for ∂_x) the arguments are straightforward.

1.2. Linearize Eq. (2) about $\phi = A_x = A_t = 0$.

1.3. Find the normal modes of this linearized equations. IE. Write $\phi(x,t) = \phi e^{ikx-i\omega t}$, and find what ω is allowed for a given k. How many independent modes do we have at each k? These correspond to particles and antiparticles (like electrons and positrons – but here they are bosonic).

As a side note, ϕ is a complex valued field so for fixed k_0, ω_0, each of the following complex exponentials are independent: $e^{i(k_0 x - \omega_0 t)}, e^{i(k_0 x + \omega_0 t)}, e^{i(-k_0 x + omega_0 t)}, e^{i(-k_0 x - \omega_0 t)}$. If ϕ was real valued things would be different: $\mathrm{re}(e^{i(k_0 x - \omega_0 t)}) = \mathrm{re}(e^{i(-k_0 x + \omega_0 t)})$.

1.4. Write the "Maxwell" equations Eq. 3 in terms of the gauge fields.

1.5. Linearize these equations about $\phi = A_x = A_t = 0$.

1.6. Write $A_t(x,t) = A_t e^{ikx-i\omega t}$ and $A_x(x,t) = A_x e^{ikx-i\omega t}$. Substitute these into the linearized equations to find a matrix equation

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} A_x \\ A_t \end{pmatrix} = 0. \tag{7}$$

Rearrange the equations so that the matrix is symmetric.

1.7. Find the eigenvalues and eigenvectors of this matrix. You should find one vector where the eigenvalue is always zero, and another where the eigenvalue is never zero.

1.8. What is the physical significance of the eigenvector with zero eigenvalue? What is the electric field E for this mode? [E is the only physical field – there is no magnetic field in 1D.]

1.9. In 3D we have 2 propegating modes for each k – these are the two transverse polarizations of the photon. In 2D we only found one – there is only one transverse direction. How many photon modes do we find in this 1D model?

Now that we understand this simple model, we are ready to study a less trivial model which illustrates the Higgs mechanism. Lets replace Eq. (2) with

$$(\partial_t - iA_t)^2\phi - (\partial_x - iA_x)^2\phi - m^2\phi + \lambda^2|\phi|^2\phi = 0. \tag{8}$$

1.10. Now linearize the Higgs equation Eq. (8) about $\phi_0 = m/\lambda$, and about $A_x, A_t = 0$. [Write $\phi = \phi_0 + a + ib$.] Take the real and imaginary parts of the equation for ϕ, so that you have two real equations. Note: you must do this separation into real and imaginary parts now, before question 1.11 where you take the Fourier transform. You will get the wrong answer if you take the Fourier transform first.

1.11. Write $a(x,t) = ae^{ikx-i\omega t}$, $b(x,t) = be^{ikx-i\omega t}$, $A_t(x,t) = A_t e^{ikx-i\omega t}$ and $A_x(x,t) = A_x e^{ikx-i\omega t}$. Substitute these into the linearized Higgs equation. One equation should decouple, the other will connect b, A_t and A_x.

1.12. Linearize the Maxwell equations about $\phi_0 = m/\lambda$, and about $A_x, A_t = 0$.

1.13. Write $a(x,t) = ae^{ikx-i\omega t}$, $b(x,t) = be^{ikx-i\omega t}$, $A_t(x,t) = A_t e^{ikx-i\omega t}$ and $A_x(x,t) = A_x e^{ikx-i\omega t}$. Substitute these into the linearized Higgs equation.

1.14. Combine the results of problem 1.11 and 1.13 as a matrix equation

$$\begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} A_x \\ A_t \\ b \end{pmatrix} = 0. \tag{9}$$

Rearrange things so that this matrix is Hermitian.

1.15. Find the eigenvalues and eigenvectors of this matrix. You should find one mode where the eigenvalue is always zero, one where the eigenvalue is never zero, and a third which has an eigenvalue which is zero when $\omega^2 = k^2 + \phi_0^2$. This last mode is the massive gauge boson (the analog of the W boson).

Problem 2. Mass of the Higgs boson
The mass of the W boson is $80GeV/c^2$.

2.1. Given our argument about the Higgs mechanism, can you say anything about the Higgs boson mass? [What is the expression for the mass of the gauge fields and the mass of the Higgs. Are there any free parameters?]

Hint: This is actually a trick question. The conventional wisdom is that you can't actually determine the mass of the Higgs from the mass of the W. Why is that so?

[Interestingly enough, the experimental mass of the Higgs is $125GeV/c^2$, which is in the same ballpark as the W mass – making one think there is actually a connection.]

Problem 3. Linear Combination of Atomic Orbitals

One way of modeling electronic structure in crystals is to imagine that the wavefunction of an electron is made from a linear combination of the atomic orbitals on each of the nuclei making up the solid. The simplest such model is to just use one orbital on each nucleus.

If r_j are the locations of each of the nuclei, then the ansatz is

$$\psi(r) = \sum_j \psi_j \phi(r - r_j), \tag{10}$$

where ψ_j are just a bunch of numbers, and $\phi(r)$ is the wavefunction of an orbital for a nucleus at the origin. [For concreteness you can imagine that $\phi(r) \sim e^{-\alpha|r|}$ is something like the $1s$ orbital of hydrogen. Regardless, we will imagine that we know $\phi(r)$ from some other calculation.]

Treating ψ as a variational wavefunction, the energy is

$$\begin{aligned} E &= \frac{\int dr\, \psi^*(r) H \psi(r)}{\int dr\, \psi^*(r)\psi(r)} && (11) \\ &= \frac{\sum_{ij} H_{ij}\psi_i^*\psi_j}{\sum_{ij} \Lambda_{ij}\psi_i^*\psi_j} && (12) \end{aligned}$$

where

$$\begin{aligned} H_{ij} &= \int dr\, \phi^*(r - r_i) H \phi(r - r_j) && (13) \\ \Lambda_{ij} &= \int dr\, \phi^*(r - r_i)\phi(r - r_j) && (14) \end{aligned}$$

3.1. Prove that H_{ij} depend on i and j only through the displacement $r_i - r_j$. Similarly, prove that Λ_{ij} also only depends on $r_i - r_j$.

3.2. What happens to H_{ij} and Λ_{ij} l when r_i and r_j are far apart?

To leading order we only need to keep diagonal terms, and terms where the two atoms are nearest neighbors:

$$\begin{aligned} H_{ij} &\approx \begin{cases} E_0 & i = j \\ -\Delta & i, j \text{ are neighbors} \\ 0 & \text{otherwise} \end{cases} && (15) \\ \Lambda_{ij} &\approx \begin{cases} 1 & i = j \\ \epsilon & i, j \text{ are neighbors} \\ 0 & \text{otherwise} \end{cases} && (16) \end{aligned}$$

Expanding to first order in the small parameters ϵ and Δ, we have

$$\begin{aligned} E &= \frac{E_0 \sum_i \psi_i^*\psi_i - t \sum_{\langle i,j\rangle} (\psi_i^*\psi_j + \psi_j^*\psi_i)}{\sum_i \psi_i^*\psi} && (17) \\ t &= \epsilon E_0 + \Delta, && (18) \end{aligned}$$

where the $\langle i, j\rangle$ indicates that we only sum over nearest neighbor pairs. Specializing to one dimension, this is

$$E = \frac{E_0 \sum_i \psi_i^*\psi_i - t \sum_i \left(\psi_{i+1}^*\psi_i + \psi_i^*\psi_{i+1}\right)}{\sum_i \psi_i^*\psi_i}. \qquad (19)$$

To optimize this wavefunction, we want to set

$$\frac{\partial E}{\partial \psi_j^*} = 0. \qquad (20)$$

Note that this is just a regular partial derivative: ψ_j^* is just an ordinary variable like x.

3.3. Prove that

$$\frac{\partial E}{\partial \psi_j^*} = \frac{-A\psi_{j+1} - A\psi_{j-1} + (E_0 - E)\psi_j}{\sum_i \psi_i^* \psi_i}, \tag{21}$$

and find A. [You may find it useful to look back at the lecture notes on the "Variational Principle" lecture, and on the "Hartree" lecture, both of which took similar derivatives.]

Setting this expression equal to zero, we are left with the eigenvalue problem

$$-A\psi_{j+1} - A\psi_{j-1} + E_0\psi_j = E\psi_j, \tag{22}$$

which is the same form as our finite difference approximation to the Schrodinger equation.

3.4. We can solve this infinite set of equations with the following ansatz:

$$\psi_j = \alpha e^{ikaj} \tag{23}$$

where a is the spacing between the nuclei, k is a parameter, and α is chosen for normalization. Plugging this ansatz into Eq. (22), find a relationship between E and k. Leave your answer in terms of A.

Problem 4. Feedback

4.1. How long did this homework take?

4.2. Which of the following words come to mind when you think about this homework (feel free to add your own words if you have something better): frustrating, fun, tedious, insightful, hard, easy, useful, useless, fair, unfair