PHYS 3317: Classical Field Theories

Friday Oct 26, 2018

Hand in by Friday November 2, 2018. We will work on these problems during recitation on Friday Oct. 26, 2018.

This week we have studied the quantum theory of sound. This is a *quantum field theory*: it is a quantum mechanical theory of a large number of oscillators, where each oscillator is at a different position in space. Today in recitation we will study a few *classical field theories.* As we learned in our previous quantum studies, much of the time the quantum system behaves identically to the classical one.

Problem 1. Dispersion Relationships

The dispersion relationship is the law which relates the angular frequency ω and the wave-vector k of plane wave excitation of a field.

1.1. Wave Equation Consider the wave equation

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) = 0. \tag{1}$$

Make the ansatz $u(x,t) = u_0 e^{ikx - i\omega t}$. Substitute this ansatz into the wave equation and find a relationship between the frequency and wave-vector.

Solution 1.1.

You should recall that the speed of a wave packet is given by $v = \partial_k \omega$. You should have found here that all waves move at the same speed. We say that the waves do not disperse – and this is the origin of the term "dispersion relationship."

1.2. Typically the quantity u in the wave equation is real (for example it might be a pressure or a displacement). Our ansatz was complex. Why was this OK, and how can we physically interpret our complex solution?

Solution 1.2.

1.3. Schrodinger Equation

Consider the Schrodinger equation

$$i\partial_t \psi(x,t) = -\frac{1}{2}\partial_x^2 \psi(x,t). \tag{2}$$

Make the ansatz $\psi(x,t) = \psi_0 e^{ikx - i\omega t}$. Substitute this ansatz into the wave equation and find a relationship between the frequency and wave-vector.

Solution 1.3.

1.4. Heat Equation

Consider the heat equation

$$\partial_t u(x,t) - \partial_x^2 u(x,t) = 0. \tag{3}$$

Make the ansatz $u(x,t) = u_0 e^{ikx - i\omega t}$. Substitute this ansatz into the wave equation and find a relationship between the frequency and wave-vector.

Solution 1.4.

1.5. How does the dispersion relationship of the Schrodinger equation differ from that of the Heat equation? In the heat equation motion is diffusive, while in the Schrodinger equation it is ballistic.

Solution 1.5.

Problem 2. Maxwell's Equations

Consider the adimensionalized Maxwell's equations in free space

$$\nabla \cdot \mathbf{E} = 0 \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{6}$$

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} \tag{7}$$

Make the ansatz

$$\mathbf{E}(\mathbf{r}, t) = [E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}} + E_z \hat{\mathbf{z}}]\, e^{ikz - i\omega t} \tag{8}$$

$$\mathbf{B}(\mathbf{r}, t) = [B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}]\, e^{ikz - i\omega t}. \tag{9}$$

2.1. From the equations $\nabla \cdot \mathbf{E}$ and $\nabla \cdot \mathbf{B} = 0$ find E_z and B_z.

Solution 2.1.

2.2. Calculate

$$\nabla \times \mathbf{E} \tag{10}$$

Solution 2.2.

2.3. Calculate

$$\nabla \times \mathbf{B} \tag{11}$$

Solution 2.3.

2.4. Calculate

$$\partial_t \mathbf{E} \tag{12}$$

Solution 2.4.

2.5. Calculate

$$\partial_t \mathbf{B} \tag{13}$$

Solution 2.5.

2.6. You should have found that Eq. (6) and (7) amount to four equations in four variables. Write these as a matrix equation

$$\begin{pmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ B_x \\ B_y \end{pmatrix} = \omega \begin{pmatrix} E_x \\ E_y \\ B_x \\ B_y \end{pmatrix} . \tag{14}$$

It will take a bit of manipulation to get it in this form: the main thing is putting the equations in the right order.

Solution 2.6.

2.7. Find the dispersion relationships for the four eigenmodes. [Hint 1: you know that because of polarization, these should be two-fold degenerate. Hint 2: The easiest way to find the eigenvalues of a matrix with lots of symmetries (like this one) is to simply guess the eigenvectors.]

Solution 2.7.

Problem 3. Nonlinear wave equations

In modeling a gas of interacting bosons, we encountered the non-linear Schrodinger equation (aka. the Gross-Pittaevskii equation):

$$i\partial_t\psi(x,t) = -\frac{1}{2}\partial_x^2\psi(x,t) - \mu\psi(x,t) + g|\psi(x,t)|^2\psi(x,t). \tag{15}$$

Here $|\psi(x,t)|^2$ is the density of particles at position x and time t. The chemical potential, μ, controls the total particle number, and g encodes the strength of interactions. A close relative of this equation also appears in superconductivity, where it is known as the time-dependent Ginzburg-Landau equation.

3.1. In equilibrium, $\psi = \sqrt{n_0}$ should be independent of space and time. Find the density, n_0, in terms of g and μ.

Solution 3.1.

3.2. To find the excitations about this homogeneous solution, we can linearize Eq. (15). We write

$$\psi(x,t) = \sqrt{n_0} + a(x,t) + ib(x,t), \tag{16}$$

where a and b are real functions. To linear order in a and b, what is

$$g|\psi(r,t)|^2\psi(r,t) \tag{17}$$

Solution 3.2.

3.3. Substitute Eq. (16) into Eq. (15), and discard terms which are quadratic or higher order, to find a homogeneous linear differential equations for a and b, with complex coefficients. Use your result from question 3.1 to eliminate μ in favor of n_0.

Solution 3.3.

3.4. Take the real and imaginary parts of your linearized equations, to produce a pair of homogeneous linear differential equations for a and b with real coefficients.

Solution 3.4.

3.5. Substitute the ansatz $a(x,t) = a_0 e^{ikx-i\omega t}$ and $b(x,t) = b_0 e^{ikx-i\omega t}$ into your equations. Rewrite your equations in the form

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \omega \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}. \tag{18}$$

Solution 3.5.

3.6. Find the dispersion relationship $\omega(k)$.

Solution 3.6.

3.7. One can expand the dispersion relationship in a power series, writing $\omega^2 = \alpha + \beta k^2 + \gamma k^4 + \cdots$. Find α and β.

Solution 3.7.

In quantum mechanics, one identifies frequencies with energies $E = \hbar\omega$ and wave-vectors with momenta, $p = \hbar k$. Relativistic particles have a dispersion relationship $E^2 = m^2c^4 + c^2p^2$. Thus one often calls $\sqrt{\alpha}$ the "mass" of the excitation, and when $\alpha = 0$ the excitations are referred to as "massless." The linearized excitations of the Gross-Pittaevskii equation are analogous to acoustic phonons, and are "massless". The optical phonons we were studying in class are "massive."

Typically photons are massless. As we will discuss next week, this is related to a symmetry: Gauge Invariance. In two weeks we will discuss the Higgs mechanism, and see how coupling between the electromagnetic field and the Gross-Pittaevskii equation can lead to massive photons. As will come up later in the course, this is one very obtuse way to think about the Meisner effect in superconductors.

The Weak Interaction is mediated by objects which are similar to photons: the W and Z bosons. Despite gauge invariance, these bosons have finite masses. We believe that the Higgs mechanism is what gives rise to their masses (and indirectly the mass of other elementary particles). To explain this result from particle physics we need to use a relativistic version of the Gross-Pittaevskii equation (the non-linear Klein-Gordon equation).