P3317 HW from Lecture 4+5 and Recitation 3

Due Tuesday September 18

Problem 1. Neutral K Mesons

The neutral K mesons are denoted K^0 and $\bar{K}^0$. [We say the "kay-zero" and the "kay-zero-bar" – the generic term for a K-meson is a kaon, pronounce "kay-on".] The K^0 is a bound state of a down quark and a strange antiquark – $d\bar{s}$. The $\bar{K}^0$ is a bound state of a strange quark and a down antiquark – $s\bar{d}$. These are "anti-particles" in the sense that a K^0 and a $\bar{K}^0$ can combine to produce photons.

The strong interaction produces K^0 and $\bar{K}^0$ particles, but they are not energy eigenstates: the weak interaction can convert a K^0 into a $\bar{K}^0$. This is analogous to the tunneling in our "double well" problem in class. Because of this interconversion, the eigenstates of the (weak) Hamiltonian are

$$K_S^0 = \frac{K^0 - \bar{K}^0}{\sqrt{2}} \tag{1}$$

$$K_L^0 = \frac{K^0 + \bar{K}^0}{\sqrt{2}}. \tag{2}$$

In words we call these the "kay-short" and "kay-long".

That is, if we think of the symbol K_S^0 representing a wavefunction, it obeys

$$HK_S^0 = E_S K_S^0 \tag{3}$$

and similarly

$$HK_L^0 = E_L K_L^0. \tag{4}$$

Both E_L and E_S are approximately 500MeV, while their difference is $E_L - E_S \sim 4 \cdot 10^{-12}$MeV.

1.1. Suppose at time $t = 0$ our kaon is described by a wavefunction

$$\psi(t = 0) = \alpha K_S^0 + \beta K_L^0, \tag{5}$$

where α and β are just some numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. By analogy to our usual rules of quantum mechanics, the probability of being in the K_S^0 state is $|\alpha|^2$, and the probability of being in the K_L^0 is $|\beta|^2$. What will the wavefunction be at a later time, t? [Hint: How do energy eigenfunctions evolve with time? By linearity you can consider each term separately. Please do not substitute numbers in – use the symbols E_L and E_S for the energies.]

Solution 1.1. (1 point) We can write $\Psi(t)$ as a superposition of K_S and K_L as follows:

$$\Psi(t) = \alpha(t)K_S + \beta(t)K_L \tag{6}$$

Substitute this into the time-dependent Schrodinger equation. Using the orthonormality of K_S and K_L, you should get the following differential equations for $\alpha(t)$ and $\beta(t)$:

$$\begin{aligned} i\hbar\frac{d\alpha(t)}{dt} &= E_S\alpha(t) \\ i\hbar\frac{d\beta(t)}{dt} &= E_L\beta(t) \end{aligned} \tag{7}$$

Solving these differential equations and plugging in the initial conditions, we get

$$\begin{aligned} \alpha(t) &= \alpha e^{-iE_St/\hbar} \\ \beta(t) &= \beta e^{-iE_Lt/\hbar} \end{aligned} \tag{8}$$

1.2. By writing K_S^0 and K_L^0 in terms of K^0 and $\bar{K}^0$, find an expression for the probability of being in the state K^0 in terms of α and β.

Solution 1.2. (1 point) We rewrite the state in the new bases as follows

$$\begin{aligned} \Psi(t) &= \alpha e^{-iE_St/\hbar}K_S + \beta e^{-iE_Lt/\hbar}K_L \\ &= \alpha e^{-iE_St/\hbar}\frac{(K-\bar{K})}{\sqrt{2}} + \beta e^{-iE_Lt/\hbar}\frac{(K+\bar{K})}{\sqrt{2}} \\ &= \frac{\alpha e^{-iE_St/\hbar} + \beta e^{-iE_Lt/\hbar}}{\sqrt{2}}K + \frac{\alpha e^{-iE_St/\hbar} - \beta e^{-iE_Lt/\hbar}}{\sqrt{2}}\bar{K} \end{aligned} \tag{9}$$

We see that the probability of being in the K state is

$$P_K = \frac{1}{2}\left|\alpha e^{-iE_St/\hbar} + \beta e^{-iE_Lt/\hbar}\right|^2 \tag{10}$$

1.3. Find an expression for the probability of being in the state $\bar{K}^0$ in terms of α and β.

Solution 1.3. (1 point) Likewise, the probability of being in the $\bar{K}$ state is

$$P_{\bar{K}} = \frac{1}{2}\left|\alpha e^{-iE_St/\hbar} - \beta e^{-iE_Lt/\hbar}\right|^2 \tag{11}$$

1.4. Suppose at time $t = 0$ a strong interaction produces a K^0. What will α and β be?

Solution 1.4. (1 point) If we are in the K state, then $\alpha = \beta = \frac{1}{\sqrt{2}}$.

1.5. At what later time will the particle have a 100% chance of being in the state $\bar{K}^0$?

Solution 1.5. (1 point) We are in the $\bar{K}$ state (up to a phase) when the coefficient of the K state is zero. From (9), we see that this occurs when

$$\frac{\alpha e^{-iE_S t/\hbar} + \beta e^{-iE_L t/\hbar}}{\sqrt{2}} = 0 \tag{12}$$

Since we saw in Part **1.4** that $\alpha = \beta$, this equation is satisfied when

$$\begin{aligned} e^{-iE_L t/\hbar} &= -e^{-iE_S t/\hbar} \\ e^{-i(E_L - E_S)t/\hbar} &= -1 \end{aligned} \tag{13}$$

This equation is satisfied if $-i(E_L - E_S)t/\hbar = (2N+1)\pi$, or $t = \frac{\hbar}{E_L - E_s}(2N+1)\pi$, where N is a nonnegative integer. The question just asked for a single time – so any of these is fine. The most important, however, is the first such time, $t = \frac{\hbar}{E_L - E_s}\pi$.

1.6. If the kaons are moving at roughly the speed of light, how far will they go before changing into their antiparticles? Give your distance in meters [neglect any relativistic time dilation].

Solution 1.6. (1 point) The first change occurs at a distance of

$$d = \frac{\hbar}{E_L - E_s}\pi c = 0.16\,\text{m}.$$

Of course, relativistic effects are important, and this is actually one of the classic demonstrations of time dilation

[As an aside, the names "long" and "short" come from the fact that the K_L^0 has a lifetime of about $5 \cdot 10^{-8}$s, while the K_S^0 has a lifetime of 10^{-11}s. As a second aside – in relativity one equates energy and mass. Thus we say that the K_L^0 and K_S^0 are the *mass* eigenstates. The K^0 and $\bar{K}^0$ are referred to as the flavor eigenstates.]

Problem 2. The rotating frame

In class we modeled the interaction of an ammonia atom with microwaves via the differential equation

$$i\partial_t \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\Delta & \epsilon\cos(\nu t) \\ \epsilon\cos(\nu t) & \Delta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \tag{14}$$

where a is the amplitude of being in the symmetric state, b the amplitude for the antisymmetric state, and ϵ is proportional to the electic field strength.

In class we studied this equation when $a \sim 0$. We found that there were separate contributions from the two exponentials making up the cosine. One of these contributions was small. As you will find out in this problem, one can exactly solve the dynamics (for any size b) if one throws away the

small contribution. In particular, we will solve the approximate problem:

$$i\partial_t \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\Delta & \epsilon e^{i\nu t}/2 \\ \epsilon e^{-i\nu t}/2 & \Delta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{15}$$

This is known as the "rotating wave approximation." It is good as long as $\nu + 2\Delta \gg \epsilon$, which is essentially always true for the physical system.

2.1. We can get rid of the time dependence in Eq. (15) by making the Ansatz:

$$\tilde{b}(t) = e^{i\nu t} b(t). \tag{16}$$

This transformation is often referred to as "going into the rotating frame."

Substitute this Ansatz into Eq. (15) to get an equation of the form

$$i\partial_t \begin{pmatrix} a \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} a \\ \tilde{b} \end{pmatrix}. \tag{17}$$

where A, B, C are all time independent. What are A, B, and C?

Solution 2.1. (2 points) We are given the equation

$$i\partial_t \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\Delta & \epsilon e^{i\nu t}/2 \\ \epsilon e^{-i\nu t}/2 & \Delta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{18}$$

We then make the ansatz $\tilde{b}(t) = e^{i\nu t} b(t)$. We note that

$$\partial_t b(t) = \partial_t(\tilde{b}(t) e^{-i\nu t}) = (\partial_t \tilde{b}(t)) e^{-i\nu t} - i\nu \tilde{b}(t) e^{-i\nu t} \tag{19}$$

We can then rewrite the given equation as in which case (19) becomes

$$i\partial_t a = -\Delta a + \frac{\epsilon}{2}\tilde{b} \tag{20}$$

$$i(\partial_t \tilde{b}) e^{-i\nu t} + \nu \tilde{b} e^{-i\nu t} = \frac{\epsilon}{2} e^{-i\nu t} a + \Delta \tilde{b} e^{-i\nu t} \tag{21}$$

We can rewrite this as

$$i\partial_t \begin{pmatrix} a \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} -\Delta & \frac{\epsilon}{2} \\ \frac{\epsilon}{2} & \Delta - \nu \end{pmatrix} \begin{pmatrix} a \\ \tilde{b} \end{pmatrix} \tag{22}$$

2.2. Find the eigenvalues λ_1, λ_2 and eigenvectors $(a_1, b_1), (a_2, b_2)$, of the matrix. What are $\lambda_1, \lambda_2, a_1, b_1, a_2, b_2$ in terms of Δ, ϵ, ν. Note that the vectors are only defined up to a multiplicative constant. You do not need to normalize the vectors (but your subsequent expressions should be consistent with your choice).

Solution 2.2. (4 points - 1 for each eigenvalue, and 1 for each eigenvector) We want to solve

$$0 = \begin{vmatrix} -\Delta - x & \frac{\epsilon}{2} \\ \frac{\epsilon}{2} & \Delta - \nu - x \end{vmatrix} = x^2 + \nu x - \Delta^2 + \nu\Delta - \frac{\epsilon^2}{4} \tag{23}$$

for x to get the eigenvalues. The solutions are

$$\lambda_1 = -\frac{\nu}{2} + \sqrt{\left(\frac{\nu}{2} - \Delta\right)^2 + \frac{\epsilon^2}{4}} \tag{24}$$

$$\lambda_2 = -\frac{\nu}{2} - \sqrt{\left(\frac{\nu}{2} - \Delta\right)^2 + \frac{\epsilon^2}{4}}. \tag{25}$$

The eigenvectors are

$$v_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2(\Delta+\lambda_1)}{\epsilon} \end{pmatrix} \text{ for } \lambda_1, \quad v_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{\epsilon}{2(\Delta+\lambda_2)} \\ 1 \end{pmatrix} \text{ for } \lambda_2 \tag{26}$$

Note that these eigenvectors are unnormalised, but it doesn't really matter if you do things consistently throughout.

2.3. Write the initial state, $a = 0, b = 1$ as the sum of these two eigenvectors. Write your answer in terms of a_1, b_1, a_2, b_2. Do **not** substitute in your expressions for these quantities in terms of Δ, ϵ, ν.

Solution 2.3. (2 points) We are given initial conditions $a(0) = 0, b(0) = \tilde{b}(0) = 1$.

$$\begin{aligned} \begin{pmatrix} a(0) \\ \tilde{b}(0) \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{a_2}{a_2 b_1 - a_1 b_2} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \frac{a_1}{a_1 b_2 - a_2 b_1} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \\ &= \frac{\epsilon}{2(\lambda_1 - \lambda_2)} v_1 - \frac{\Delta + \lambda_2}{\lambda_1 - \lambda_2} v_2 \end{aligned} \tag{27}$$

2.4. What is $a(t)$ and $\tilde{b}(t)$? Write your answer in terms of $a_1, b_1, a_2, b_2, \lambda_1, \lambda_2$. Do **not** substitute in your expressions for these quantities in terms of Δ, ϵ, ν.

Solution 2.4. (2 points) We can rewrite our state in the eigenvector basis:

$$\begin{pmatrix} a(t) \\ \tilde{b}(t) \end{pmatrix} = c_1(t)v_1 + c_2(t)v_2 \tag{28}$$

(22) then becomes

$$i\partial_t c_1(t)v_1 + i\partial_t c_2(t)v_2 = \lambda_1 c_1(t)v_1 + \lambda_2 c_2(t)v_2 \tag{29}$$

We hence obtain differential equations for c_1 and c_2, with general solutions

$$c_1(t) = c_1(0)e^{-i\lambda_1 t}, \quad c_1(t) = c_2(0)e^{-i\lambda_2 t} \tag{30}$$

Given the initial conditions $c_1(0)$ and $c_2(0)$ we found in (27), we have:

$$\begin{aligned} \begin{pmatrix} a(t) \\ \tilde{b}(t) \end{pmatrix} &= \frac{a_2}{a_2b_1 - a_1b_2}e^{-i\lambda_1 t}v_1 + \frac{a_1}{a_1b_2 - a_2b_1}e^{-i\lambda_2 t}v_2 \\ &= \frac{\epsilon}{2(\lambda_1 - \lambda_2)}e^{-i\lambda_1 t}v_1 - \frac{\Delta + \lambda_2}{\lambda_1 - \lambda_2}e^{-i\lambda_2 t}v_2 \end{aligned} \tag{31}$$

2.5. If we are on resonance, $\nu = 2\Delta$, everything simplifies. Write $a(t)$ and $\tilde{b}(t)$ for this case [in terms of Δ and ϵ].

Solution 2.5. (2 points) If we are on resonance,

$$\lambda_1 = -\Delta + \frac{\epsilon}{2} \tag{32}$$

$$\lambda_2 = -\Delta - \frac{\epsilon}{2} \tag{33}$$

Then (31) becomes

$$\begin{pmatrix} a(t) \\ \tilde{b}(t) \end{pmatrix} = \frac{1}{2}e^{-i\lambda_1 t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^{-i\lambda_2 t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{34}$$

So we have

$$a(t) = -e^{i\Delta t}i\sin(\frac{\epsilon}{2}t) \tag{35}$$

$$\tilde{b}(t) = e^{i\Delta t}\cos(\frac{\epsilon}{2}t) \tag{36}$$

2.6. If we are on resonance, $\nu = 2\Delta$, there will be a time when the molecule has a 100% chance of being in the symmetric state. What is the earliest such time? This is the ideal amount of time that the molecule should be in the cavity.

Solution 2.6. (2 points) We are in the symmetric state when $\tilde{b}(t) = 0$. This occurs when $\cos(\frac{\epsilon}{2}t) = 0$, or $t = \frac{\pi}{\epsilon}$ at the first instance.

2.7. In recitation 3 you numerically solved Eq. (14) – though you did it in a different basis. Rewrite your code to solve Eq. (14) in the basis given.

Compare your numerical solution to your analytic "Rotating Wave" approximation. In particular, take the resonant case $\nu = 2\Delta$, with $\Delta = 1$ and $\epsilon = 0.1$. Make a properly labeled plot which shows the probability of being in the symmetric state as a function of time – including both the approximate and numerically exact solution. Choose your step size for the numerics sufficiently small that there is no noticable error from the finite step size. Choose the scale of your graph so that it includes a bit more than one "Rabi flop" (that is, choose the time interval so that the probability of being in the symmetric state becomes maximal, and starts dropping again). Include a legend.

Upload your notebook to Blackboard – but also include a print-out of your graph with the homework sheets that you hand in.

Solution 2.7. (1 point for uploading notebook – grader will not go through notebook though) One way to produce the plots is to use:

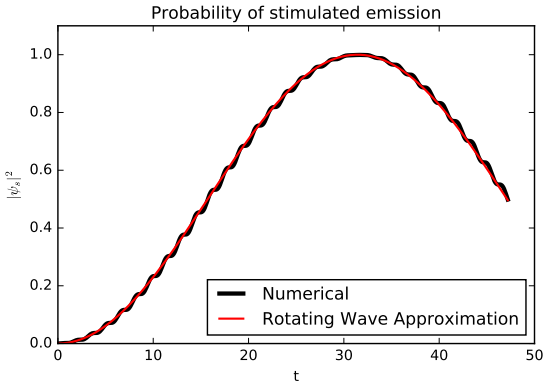
```
%pylab inline
def homeworkH(t,delta=1,eps=0.1,nu=2):
    mat=array([[-delta,eps*cos(nu*t)],[eps*cos(nu*t),delta]])
    return mat

def homeworkU(t,dt=0.1,eps=0.1,delta=1,nu=2):
    mat1=eye(2) -1.j*(dt/2)*homeworkH(t=t,delta=delta,eps=eps,nu=nu)
    mat2=eye(2) +1.j*(dt/2)*homeworkH(t=t,delta=delta,eps=eps,nu=nu)
    U=inv(mat2).dot(mat1)
    return U
```

Solution 2.7. continued...

```
def hwtimeseries(maxt,dt,eps=0.1):
    t=0
    psi=array([0.,1.])
    psilist=[psi]
    tlist=[t]

    while (t<maxt):
        t=t+dt
        evolve=homeworkU(t,dt,eps)
        psi=evolve.dot(psi)
        tlist.append(t)
        psilist.append(psi)

    psiarray=array(psilist)
    tarray=array(tlist)

    return {"psi":psiarray,"t":tarray,"eps":eps,"dt":dt,"maxt":maxt}

ser1=hwtimeseries(maxt=1.5*pi/0.1,dt=0.1,eps=0.1)

psilist=ser1["psi"]
plist=[abs(psi[0])**2 for psi in psilist]
tlist=ser1["t"]
analytic=sin(0.1*tlist/2)**2

plot(tlist,plist,linestyle="-",linewidth=3,color="k", label="Numerical")
plot(tlist,analytic,"r",linewidth=1.5,label="Rotating Wave Approximation")
ylim(0,1.1)
xlabel("t")
ylabel("$|\psi_s|^2$")
title("Probability of stimulated emission")
legend(loc="lower right")
savefig("se.pdf")
```

Solution 2.7. continued...

This gives something like:
(5 points for graph – 1 for labels, 1 for choosing scale, 2 for numerical result, 1 for analytic result)



2.8. Describe in words the difference between the results you found with the analytic approximation and the numerics. What features are captured well by the approximation, and which are missed?

Solution 2.8. (2 points) The general shape, and even the time-scale is well modeled by the rotating wave approximation. What is missed is high frequency "wiggles." This makes sense, as we neglected some highly oscillatory terms.

Problem 3. Write a matrix that corresponds to a finite difference approximation to the Harmonic Oscillator Hamiltonian. Explain your notation.

Solution 3.1. (4 points – 2 for getting derivative matrix right, 2 points for the potential matrix) The Hamiltonian is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 \tag{37}$$

A finite difference version of the second derivative (central second derivative) reads

$$f_n'' = \frac{f_{n+1} - 2f_n + f_{n-1}}{\Delta x^2} \tag{38}$$

Let us use the indexing set $\{0, 1, ...nmax\}$ for our grid. As discussed in the recitations, we can refine the corresponding matrix in a few ways. One way is to use periodic boundary conditions, i.e. we impose $f(x_{-1}) = f(x_{nmax})$ and $f(x_{nmax+1}) = f(x_0)$. We can then write the Hamiltonian as

$$H = -\frac{\hbar^2}{2m\Delta x^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix} + \frac{1}{2}m\omega^2 \begin{pmatrix} x_0^2 & 0 & \cdots & 0 \\ 0 & x_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{nmax}^2 \end{pmatrix} \tag{39}$$

where $x_n = x_0 + n\Delta x$. We note that the 1's in the top-right and bottom-left corners of the first term are due to the choice of periodic boundaries. You are welcome to use other boundary conditions, in which case those 1's will not be there.

Another acceptable version of the kinetic energy matrix is

$$K = -\frac{\hbar^2}{8m\Delta x^2}\begin{pmatrix} -2 & 0 & 1 & \cdots & 0 & 0 \\ 0 & -2 & 0 & 1 & \cdots & 0 \\ 1 & 0 & -2 & 0 & 1 & \cdot \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 1 & 0 & -2 \end{pmatrix}. \tag{40}$$

There are some subtle numerical issues with this choice, but it will generally work. Note the factor in the denominator, and this time I used hard wall boundary conditions.

Problem 4. Feedback

4.1. How long did this homework take?

4.2. Which of the following words come to mind when you think about this homework (feel free to add your own words if you have something better): frustrating, fun, tedious, insightful, hard, easy, useful, useless, fair, unfair