P3317 HW from Lecture 7+8 and Recitation 4

Due Friday Tuesday September 25

Problem 1. In class we argued that an ammonia atom in an electric field can be modeled by a two-level system, described by a Schrodinger equation

$$i\hbar\partial_t \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} E_0 + \epsilon & -\Delta \\ -\Delta & E_0 - \epsilon \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \tag{1}$$

The constant E_0 plays no role in the dynamics: this is supposed to represent the energy when the Nitrogen atom is at one of its two potential minima. The term Δ accounts for tunneling, and ϵ is proportional to the electric field applied along the symmetry axis of the molecule.

Experiments show good agreement with this model, and the most accurate way to determine ϵ and Δ is not from a theoretical calculation, but by fitting the prediction of this model to experiment. One finds $\Delta = 10^{-4}$eV, and $\epsilon = \mu\mathcal{E}$, where $\mathcal{E}$ is the electric field strength. The proportionality constant is the electric dipole moment, $\mu = 1.6$ debye. A debye is the CGS unit for electric dipole moments: 1 debye= 3.3×10^{-30}Cm. A debye is a typical electric dipole moment for a molecule – it is equal to the charge of the electron times a distance of roughly 0.39 Bohr.

1.1. Derive an expression for the energy eigenvalues as a function of E_0, Δ, and ϵ. Make a sketch of Energy vs ϵ for fixed E_0 and Δ. Please do not substitute in numbers – I just want to know the shape of the curve. Please do label the axes though. Unlike "graphing" problems, a hand-drawn sketch is fine here (though a computer generated one is also fine).

As an aside, you should see that one state is "high field seeking" – meaning its energy is lowest in regions of large field, while the other is "low field seeking". This property is what is used for creating an inverted population for an ammonia Maser.

Solution 1.1. (2 points for eigenvalues, 2 points for sketch)
The Hamiltonian matrix:

$$H = \begin{pmatrix} E_0 + \epsilon & -\Delta \\ -\Delta & E_0 - \epsilon \end{pmatrix} \tag{2}$$

has eigenvalues given by the determinant equation:

$$\begin{vmatrix} E_0 + \epsilon - \lambda & -\Delta \\ -\Delta & E_0 - \epsilon - \lambda \end{vmatrix} = 0 \tag{3}$$

$$\lambda^2 - 2\lambda E_0 + E_0^2 - \epsilon^2 - \Delta^2 = 0 \tag{4}$$

$$\implies \lambda_\pm = E_0 \pm \sqrt{\epsilon^2 + \Delta^2}. \tag{5}$$

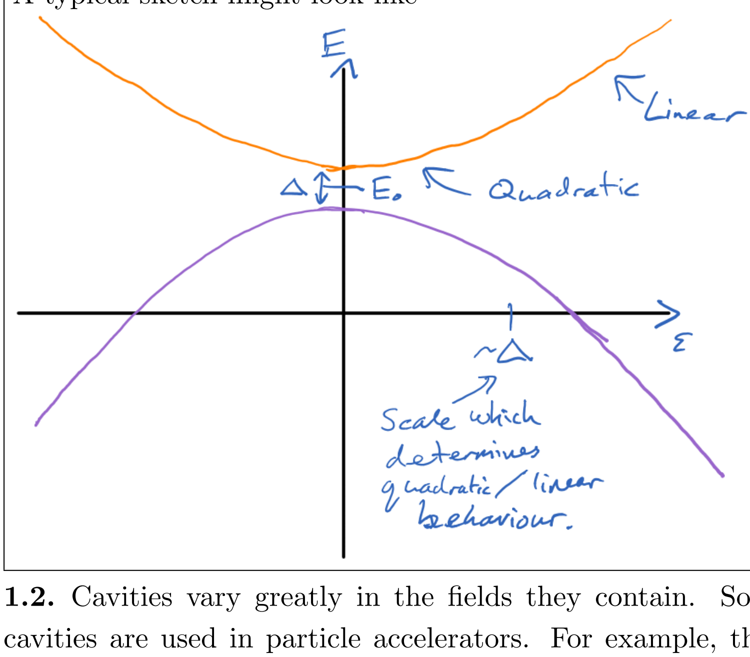
In order to make a sketch of this, it is useful to find the asymptotic forms of the expression in the large and small ϵ limits. These are:

$$\epsilon \ll \Delta : \quad \lambda_\pm = E_0 \pm \Delta\sqrt{1 + \frac{\epsilon^2}{\Delta^2}} \tag{6}$$

$$\simeq E_0 \pm \left(\Delta + \frac{\epsilon^2}{2\Delta}\right) \tag{7}$$

$$\epsilon \gg \Delta : \quad \lambda_\pm \simeq E_0 \pm \epsilon. \tag{8}$$

So for small ϵ, the energy eigenvalues are quadratic functions of ϵ with positive and negative curvature; for large ϵ, they become linear functions with positive and negative gradient. The average energy is controlled by E_0, with an energy splitting controlled by Δ. The function turns from quadratic to linear at a scale controlled by Δ.
A typical sketch might look like



1.2. Cavities vary greatly in the fields they contain. Some of the highest intensity microwave cavities are used in particle accelerators. For example, the next generation "Energy Recovering Linac" being built at Cornell has superconducting cavities with electric fields as large as 50 MV/m. For these extremely high-intensity cavities, how does ϵ compare with Δ? [More concretely, what is the ratio ϵ/Δ?] Would the perturbation theory we did in class be appropriate for these cavities? The original MASER experiments had much smaller fields.

Solution 1.2. The perturbation theory depends on small ϵ/Δ. When this ratio is small, a power series in it converges quickly. When it approaches $\mathcal{O}(1)$, the perturbative expansion begins to break down. For $\mathcal{E} = 50$ MV/m, we have:

$$\begin{aligned} \frac{\epsilon}{\Delta} &= \frac{\mu\mathcal{E}}{\Delta} \\ &= \frac{(1.6 \times 3.3 \times 10^{-30}\ \mathrm{Cm})(50\ \mathrm{MV/m})}{10^{-4}\ \mathrm{eV}} \\ &= 2.64 \times 10^{-24} \left(\frac{\mathrm{C\ MV}}{\mathrm{eV}}\right) \\ &= 2.64 \times 10^{-24} \left(\frac{C}{e}\frac{MV}{V}\right) \\ &= 2.64 \times 10^{-24} \times \frac{1}{1.6 \times 10^{-19}} \times 10^{6} \\ &= 17. \end{aligned} \tag{9}$$

Since this ratio is actually much greater than 1, the perturbation theory breaks down. Note that these electric fields are much larger than you would find in a typical maser.

Note, in the past I have found that some students do not have a very systematic approach to unit conversion. My *strong* recommendation is to keep all units at all stages of your calculation – and do the conversions by multiplying and dividing by ratios which equal 1. For example, if I wanted to convert 60 miles per hour into meters per second, I would look up on google that there are 1609 meters in 1 mile, and write

$$v = \frac{60\,\mathrm{miles}}{\mathrm{hour}} \times \frac{1609\,\mathrm{m}}{\mathrm{mile}} \times \frac{1\,\mathrm{hour}}{60\,\mathrm{minutes}} \times \frac{1\,\mathrm{minute}}{60\,\mathrm{s}} = 27\,\mathrm{m/s}. \tag{10}$$

In fact, any time I am working with dimensional quantities, I write down the units, and carry them through all of the arithmetic. I can't tell you how many times I have tracked down an error by finding that the units don't work out properly. It makes me cringe when (even as an intermediate step) I see someone write something like $v = 3$, when v is supposed to be dimensional. Of course I am happy when students first explicitly adimensionalize their equations, then work simply with numbers.

Problem 2.

2.1. Two-level Hamiltonians, are often described in terms of 2×2 matrices

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (11)$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (12)$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (13)$$

Are these Hermitian, Unitary, both?

Solution 2.1. (2 points)

1. Each of these matrices satisfies the property $\sigma^\dagger = \sigma$, and therefore they are Hermitian.

2. Each of these matrices satisfies the property $\sigma^\dagger \sigma = \mathbf{1}$, and so they are Unitary.

2.2. What are the eigenvalues of the Hamiltonian

$$H = E_0 \mathbf{1} + B_x \sigma_x + B_y \sigma_y + B_z \sigma_z, \quad (14)$$

where $\mathbf{1}$ is the identity matrix? No need to find the eigenvectors, I just want the eigenvalues as a function of E_0 and the vector B_x, B_y, B_z. This is a good result to remember (and I often calculate the eigenvalues of 2x2 matrices by writing them in this form).

Solution 2.2. (2 points)
Writing out H as a matrix, we have:

$$H = \begin{pmatrix} E_0 + B_x & B_y - iB_z \\ B_y + iB_z & E_0 - B_x \end{pmatrix}, \quad (15)$$

and so the eigenvalues are given by the solution to:

$$\begin{vmatrix} E_0 + B_x - \lambda & B_y - iB_z \\ B_y + iB_z & E_0 - B_x - \lambda \end{vmatrix} = 0, \quad (16)$$

$$\lambda^2 - 2\lambda E_0 + E_0^2 - |\mathbf{B}|^2 = 0, \quad (17)$$

$$\implies \lambda_\pm = E_0 \pm |\mathbf{B}|. \quad (18)$$

The effect of the vector $\mathbf{B}$, which might for instance be an applied magnetic field on a spin 1/2 particle (up to a dimensionful factor), is to split the degenerate energy levels of the original Hamiltonian $H = E_0 \mathbf{1}$.

2.3. The notation $\sigma_x, \sigma_y, \sigma_z$ comes from the fact that in the case of *spin*, the object $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ transforms under rotation as a vector. Feynman Lectures volume 3, chapter 6 works you through the logic that proves it. Furthermore, we can interpret the positive eigenvectors of Eq. (14) as a quantum state in which the spin is pointing in the direction of $\tilde{\mathbf{B}} = (B_x, B_y, B_z)$. You do not need to prove this, but it turns out that you can write this eigenstate as

$$|\hat{n}\rangle = \begin{pmatrix} \cos(\theta/2)e^{i\phi/2} \\ \sin(\theta/2)e^{-i\phi/2} \end{pmatrix}, \tag{19}$$

where $\tan\theta = \sqrt{B_x^2 + B_y^2}/B_z$ and $\tan\phi = B_y/B_x$ describe the direction of the vector $\mathbf{B}$. For convenience we label the state with the unit vector $\hat{n} = \tilde{\mathbf{B}}/|B|$.

What is the $+z$-eigenstate, $|\hat{z}\rangle$? [You can use Eq. (19), or go back to Eq. (14) – either is fine.]

Solution 2.3. (1 point)
Substituting in $\theta = 0$ we have

$$|\hat{z}\rangle = e^{i\phi}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{20}$$

The choice of ϕ is arbitrary and carries no physical meaning. You are free to choose any value you want for it – the simplest is $\phi = 0$.

2.4. What is the $+x$-eigenstate, $|\hat{x}\rangle$? [You can use Eq. (19), or go back to Eq. (14) – either is fine.]

Solution 2.4. (1 point)
Substituting in $\theta = \pi/2$ and $\phi = 0$ yields

$$|\hat{x}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{21}$$

2.5. What is the $+y$-eigenstate, $|\hat{y}\rangle$? [You can use Eq. (19), or go back to Eq. (14) – either is fine.]

Solution 2.5. (1 point)
Substituting in $\theta = \pi/2$ and $\phi = \pi/2$ yields

$$|\hat{y}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1+i}{\sqrt{2}} \\ \frac{1-i}{\sqrt{2}} \end{pmatrix} \tag{22}$$

$$= \frac{1+i}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{23}$$

2.6. In class we introduced the notation $\langle\psi|\phi\rangle$ for the dot-product between two vectors in an abstract Hilbert space. In particular, if

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{24}$$

$$|\phi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{25}$$

then we define

$$\langle\psi|\phi\rangle = \psi_1^*\phi_1 + \psi_2^*\phi_2. \tag{26}$$

Calculate the overlap $\langle\hat{x}|\hat{z}\rangle$.

2.7. Find the overlap $\Lambda = \Big\langle \cos(\theta_1)\hat{z} + \sin(\theta_1)\hat{x} \Big| \cos(\theta_2)\hat{z} + \sin(\theta_2)\hat{x} \Big\rangle$. Show that Λ is only a function of $\delta\theta = \theta_1 - \theta_2$.

This is a special case of a more general relation that $|\langle\hat{n}|\hat{n}'\rangle|$ is only a function of the angle between the two unit vectors $\hat{n}$ and $\hat{n}'$.

Solution 2.6. (2 points)

$$\begin{aligned}
\Lambda &= \begin{pmatrix} \cos(\theta_1/2) & \sin(\theta_1/2) \end{pmatrix} \begin{pmatrix} \cos(\theta_2/2) \\ \sin(\theta_2/2) \end{pmatrix} && (27)\\
&= \cos(\theta_1/2)\cos(\theta_2/2) + \sin(\theta_1/2)\sin(\theta_2/2) && (28)\\
&= \cos\left(\frac{\theta_1 - \theta_2}{2}\right). && (29)
\end{aligned}$$

Problem 3. In the next problem we will use the following integral

$$I = \int dx \frac{\sin^2 x}{x^2} = \pi. \tag{30}$$

Here you will verify that result using a trick called "integrating under the integral." [Apparently Feynman was fond of tricks like this.]

3.1. By integrating over t_1 and t_2, show

$$I = \frac{1}{4}\int_{-\infty}^{\infty} dx \int_{-1}^{1} dt_1 \int_{-1}^{1} dt_2\, e^{ix(t_1+t_2)} \tag{31}$$

Solution 3.1. (2 points)
We begin by noting that

$$\int_{-1}^{1} dt\, e^{ixt} = \frac{e^{ix} - e^{-ix}}{ix} = \frac{2\sin(x)}{x}. \tag{32}$$

Thus

$$\begin{aligned}
\int_{-1}^{1} dt_1 \int_{-1}^{1} dt_2\, e^{ix(t_1+t_2)} &= \left(\int_{-1}^{1} dt\, e^{ixt}\right)^2 && (33)\\
&= 4\frac{\sin^2 x}{x^2}. && (34)
\end{aligned}$$

Thus

$$I \equiv \int dx\, \frac{\sin^2 x}{x^2} = \frac{1}{4}\int_{-\infty}^{\infty} dx \int_{-1}^{1} dt_1 \int_{-1}^{1} dt_2\, e^{ix(t_1+t_2)}. \tag{35}$$

3.2. Perform the x integral in EQ. (31), using the identity

$$\int_{-\infty}^{\infty} dx\, e^{ixs} = 2\pi\delta(s). \tag{36}$$

Solution 3.2. (1 point)
Rearranging the order order of the integrals in Eq. (**??**), we get:

$$I = \frac{1}{4}\int_{-1}^{1} dt_1 \int_{-1}^{1} dt_2 \int_{-\infty}^{\infty} dx\, e^{ix(t_1+t_2)}, \tag{37}$$

and using the identity given, the inner integral gives:

$$I = \frac{\pi}{2}\int_{-1}^{1} dt_1 \int_{-1}^{1} dt_2\, \delta(t_1 + t_2). \tag{38}$$

3.3. Now the t_1 and t_2 integrals should now be straightforward. Perform them.

Solution 3.3. (2 points)
The quick way to evaluate this integral is directly:

$$\int_{-1}^{1} dt_2\, \delta(t_1 + t_2) = 1 \qquad\qquad \text{if } -1 < t_2 < 1 \tag{39}$$

$$\implies I = \frac{\pi}{2} \int_{-1}^{1} dt_1 \tag{40}$$

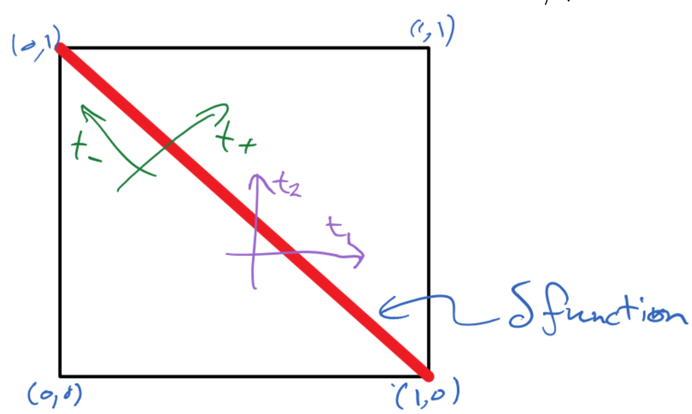
$$= \pi. \tag{41}$$

If you found this result unintuitive (that the integral over the delta function gave 2 rather than 1), consider the integral in a rotated basis: $t_\pm = (t_1 \pm t_2)/\sqrt{2}$. Then the integral is:

$$\int_{-1}^{1} dt_1 \int_{-1}^{1} dt_2\, \delta(t_1 + t_2) = \int_{-\sqrt{2}}^{\sqrt{2}} dt_- \int_{-a(t_-)}^{a(t_-)} dt_+\, \delta\left(\sqrt{2} t_+\right) \tag{42}$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} dt_- \frac{1}{\sqrt{2}}. \tag{43}$$

Pictorially, in the first approach we are integrating over the sides of a square, in which the delta function is along a diagonal. In the second approach, the integration region has the shape of a diamond and so happens more complicated, but the delta function is along one of the integration directions and it has cross sectional area $1/\sqrt{2}$ when it is cut in this direction:



Problem 4. Although we did not spend much time exploring it, in class we showed that the probability of an ammonia atom absorbing radiation of detuning δ in time t is

$$P = 4\epsilon^2 t^2 \frac{\sin^2(\delta t/2)}{(\delta t/2)^2}, \tag{44}$$

where ϵ is proportional to the electric field (so ϵ^2 is proportional to the intensity of the light, I_0.) Imagine we have many ammonia atoms, and light of many frequencies. The total energy absorbed from the light in that time is then

$$E \propto I_0 t^2 \int d\delta \frac{\sin^2(\delta t/2)}{(\delta t/2)^2}. \tag{45}$$

Use the identity

$$\int dx \frac{\sin^2 x}{x^2} = \pi \tag{46}$$

to show that the absorbed **power** is a constant, independent of time.

Solution 4.1. (2 points)
The total energy absorbed during time t is

$$E \propto I_0 t^2 \int d\delta \frac{\sin^2(\delta t/2)}{(\delta t/2)^2} \tag{47}$$

$$= 2 I_0 t \pi. \tag{48}$$

The power, $P = \partial E/\partial t$, is clearly independent of time.

Problem 5. Here we will use dimensional analysis to estimate the ionization energy of hydrogen. The various quantities which come to mind as being involved are: The ionization energy E_0, the proton mass m_p, electron mass m_e, Coulomb's constant $k = e^2/(4\pi\epsilon_0)$, planck's constant h (or equivalently $\hbar = h/2\pi$. The physics is non-relativistic, so we don't expect to need c.

5.1. How many independent dimensionless quantities can you construct from these parameters? Write them down.

Solution 5.1 (3 points)**.** In SI units, the given quantities have dimension:

$$[E_0] = \text{kg m}^2 \text{ s}^{-2}, \quad [m_p] = [m_e] = \text{kg}, \quad [k] = \text{kg m}^3 \text{ s}^{-2}, \quad [\hbar] = \text{kg m}^2 \text{ s}^{-1}. \tag{49}$$

One way to keep track of this is to remember that [force] = kg m s^2, [energy] = [force $\times$ distance], and $[\hbar]$ = [energy $\times$ time]. By combining powers of these dimensionful quantities we can make dimensionless quantities:

$$\Pi_n = E_0^a m_p^b m_e^c k^d \hbar^e, \tag{50}$$

if the powers a, b, c, d, e satisfy the linear equations:

$$\begin{aligned} &\text{kg:} & a - b - c + d + e &= 0, & (51)\\ &\text{m:} & 2a + 3d + 2e &= 0, & (52)\\ &\text{s:} & -2a - 2d - e &= 0. & (53) \end{aligned}$$

It is probably easier to find the dimensionless combinations by inspection rather than by solving these equations, but writing them down is helpful in determining how many independent dimensionless quantities we can construct. These are three equations in five unknowns. Assuming no special degeneracies, this will leave two degrees of freedom unspecified. Thus one expects that the most general solution will have two unknowns $\Pi_1^{\alpha_1}\Pi_2^{\alpha_2}$, corresponding to two independent dimensionless parameters. A convenient choice for these parameters are:

$$\Pi_0 = \frac{m_e}{m_p}, \qquad \Pi_1 = \frac{E_0}{m_e}\frac{\hbar^2}{k^2}. \tag{54}$$

Any two quantities which are powers of these, or products of powers of these is a valid answer.

5.2. What is the most general expression for E_0 which only involves these scales, and is dimensionally consistent.

Solution 5.2. (2 points)
Since we only have two dimensionless quantities describing the system, they must have a relationship of the form:

$$\Pi_1 = f(\Pi_0), \tag{55}$$

for some function f. It follows that:

$$\begin{aligned} \frac{E_0}{m_e}\frac{\hbar^2}{k^2} &= f\left(\frac{m_e}{m_p}\right)\\ \implies E_0 &= \frac{m_e k^2}{\hbar^2} f\left(\frac{m_e}{m_p}\right). \end{aligned} \tag{56}$$

An equally valid, but less systematic way of coming to this conclusion is to observe that of the quantities $m_p, m_e, k, \hbar$, there is only one way of forming a quantity with dimensions of energy up to ratios of m_e/m_p.

5.3. If we assume that m_p drops out, what do you find for the ionization energy? [Note, that as with any other dimensional argument, all you get is an order of magnitude.] Give a number in eV.

Solution 5.3. (2 points)
If m_p drops out (which might be expected in the limit $m_e/m_p \to 0$), then we have that:

$$E_0 = n\frac{m_e k^2}{\hbar^2}, \tag{57}$$

where n is some dimensionless number that we would generically expect to be $\mathcal{O}(1)$. See the aside below for some discussion on this. Therefore, we expect to get an energy:

$$\begin{aligned} E_0 &\sim \frac{9\times 10^{-31}\text{ kg} \times ((2\times 10^{-19})^2 \times 9\times 10^{9}\text{ kg m}^3\text{ s}^{-2})^2}{(7\times 10^{-16}\text{ eV s})^2}, \\ &\sim \frac{16}{49}\times 10^{-55}\times \frac{(\text{kg m}^2\text{ s}^{-2})^3}{\text{eV}^2} \\ &\sim \frac{16}{49}\times 10^{-55}\times \frac{1}{(1.6\times 10^{-19})^3}\text{ eV} \\ &\sim \frac{4}{49}\times 10^{2}\text{ eV} \sim 10\text{ eV}. \end{aligned} \tag{58}$$

The ionization energy of Hydrogen is 13.6 eV, so this was a good estimate!

Aside: $\mathcal{O}(1)$ Dimensionless Coefficients

In Eq. (57), we had an unknown dimensionless coefficient n. It might be the solution to some dimensionless differential or integral equation. Most of the time, these give $\mathcal{O}(1)$ numbers, and the dimensional analysis works. In those cases that it does not work, and the dimenesionless coefficient is incredibly tiny or huge, it is usually associated with some interesting physics. For example, in some random physics problem the dimensionless number might be coming from an integral like:

$$n \sim \int_{-\infty}^{\infty} e^{-x^2} dx \sim 1. \tag{59}$$

If, on the other hand, it was coming from:

$$n \sim \int_{-\infty}^{\infty} e^{-e^{e^{e^{x^2}}}} dx \sim 10^{-8} \ll 1, \tag{60}$$

then our dimensional analysis estimate would have been quite wrong. But a physical system that gave you an integral like this would be very strange and very rare indeed.

Problem 6. The Schrodinger equation for a simple harmonic oscillator reads

$$-\frac{\hbar^2}{2m}\partial_x^2\psi(x) + \frac{1}{2}m\omega^2 x^2\psi(x) = E\psi(x). \tag{61}$$

Rescale the variables so that $y = x/x_0$ and $\mathcal{E} = E/E_0$ to reduce this to the dimensionless equation

$$-\frac{1}{2}\partial_y^2\psi + \frac{1}{2}y^2\psi = \mathcal{E}\psi. \tag{62}$$

What is x_0 and E_0?

Solution 6.1. (4 points – 2 for x_0 and 2 for E_0)
The original forma of the question had a typo, the energy rescaling equation should have been $\mathcal{E} = E/E_0$, and I shall use this definition for the solution. You will get full marks if you did the original version (in which case $E_0 \to 1/E_0$).
Making the substitutions, we get:

$$-\frac{\hbar^2}{2m}\frac{1}{x_0^2}\partial_y^2\psi(y) + \frac{1}{2}m\omega^2 x_0^2 y\psi(y) = \mathcal{E}E_0\psi(y). \tag{63}$$

We therefore want to choose E_0 and x_0 such that:

$$\frac{\hbar^2}{mx_0^2} = m\omega^2 x_0^2 = E_0. \tag{64}$$

This is two equation in two unknowns, so should be solvable. One pair of equations gives:

$$x_0 = \left(\frac{\hbar}{m\omega}\right)^{1/2}, \tag{65}$$

then solving for E_0 gives:

$$E_0 = \hbar\omega. \tag{66}$$

This will give us the desired form of the equation.

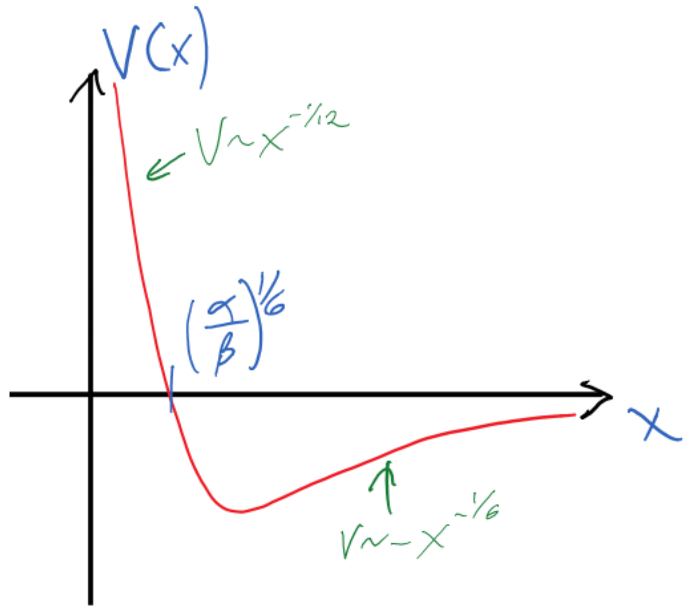
Problem 7. A typical model for the potential between two atoms is

$$V = \frac{\alpha}{x^{12}} - \frac{\beta}{x^6}, \tag{67}$$

where α and β are constants which depend on the atom. This is often known as a “6-12” potential.

7.1. Sketch this potential.

Solution 7.1. (2 points)
For sufficiently small x, the x^{-12} term will dominate, giving $V \sim \alpha/x^{12}$. For sufficiently large x, the negative term will dominate, giving $V \sim -\beta/x^6$. Since this function goes from positive to negative, it will cross the x axis, which happens at $x^6 = \alpha/\beta$. This is sufficient information to sketch the potential:



7.2. What are the units of α and β?

Solution 7.2. (2 points)
Since the potential has dimensions of energy, the parameters α and β have units:

$$[\alpha] = \text{J m}^{12} \qquad [\beta] = \text{J m}^{6}. \tag{68}$$

It is also reasonable to interpret this potential as the potential energy per unit charge, in which case J is to be replaced by V.

7.3. There is a unique length-scale r_0 you can make out of α and β. What is r_0?

Solution 7.3. (2 points)
We need it to be of the form $(\alpha/\beta)^n$, in order to cancel out the J's, and then the power n is determined by the requirement that there is one power of distance:

$$r_0 = \left(\frac{\alpha}{\beta}\right)^{1/6} \tag{69}$$

7.4. Since there is only one length-scale here, the location of the minimum of V, must roughly be given by r_0. Use calculus to find the location of the minimum r^*, and calculate r^*/r_0.

Solution 7.4. (2 points)
The standard procedure is to solve $V' = 0$:

$$\begin{aligned} \frac{dV}{dx}\Big|_{r^*} &= -12\frac{\alpha}{r^{*13}} + 6\frac{\beta}{r^{*7}} = 0 \\ \implies r^* &= 2^{1/6}\, r_0. \end{aligned} \tag{70}$$

In decimals, $r^*/r_0 \simeq 1.1$.

Problem 8. Feedback

8.1. How long did this homework take?

8.2. Which of the following words come to mind when you think about this homework (feel free to add your own words if you have something better): frustrating, fun, tedious, insightful, hard, easy, useful, useless, fair, unfair