P3317 HW from Lecture 8+9 and Recitation 5

Due Tuesday October 2

Problem 1. Two spin 1/2 particles
The space of states describing two spin 1/2 particles is spanned by four states, which are conveniently written,

$$\begin{aligned} |\uparrow\uparrow\rangle &= |\uparrow\rangle \otimes |\uparrow\rangle && (1)\\ |\uparrow\downarrow\rangle &= |\uparrow\rangle \otimes |\downarrow\rangle && (2)\\ |\downarrow\uparrow\rangle &= |\downarrow\rangle \otimes |\uparrow\rangle && (3)\\ |\downarrow\downarrow\rangle &= |\downarrow\rangle \otimes |\downarrow\rangle. && (4) \end{aligned}$$

In each of these states, the spin projection of each of the spins has a definite value in the $\hat{z}$ direction.

An important set of operators on this space are those which are a product of something happening to each spin. For example $\sigma_z \otimes \sigma_x$ is the operator which simultaneously acts as a Pauli σ_z operator on the first spin, and a Pauli σ_x on the second. Sometimes this is instead written $\sigma_{z1}\sigma_{x2}$. For example,

$$\sigma_z \otimes \sigma_x |\downarrow\downarrow\rangle = -|\downarrow\uparrow\rangle. \tag{5}$$

The "$-$" sign comes from the σ_z on the first spin. The σ_x then flips the second spin.

1.1. Consider the action of $\sigma_z \otimes \sigma_x$ on the basis vectors,

$$\sigma_z \otimes \sigma_x \begin{pmatrix} |\uparrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{pmatrix} = \begin{pmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix} \begin{pmatrix} |\uparrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{pmatrix}. \tag{6}$$

Call the matrix on the right $\bar{M}$. Find $\bar{M}$.

Solution 1.1. (2 points)
The matrix elements are

$$\begin{aligned}
\sigma_z \otimes \sigma_x | \uparrow\uparrow\rangle &= | \uparrow\downarrow\rangle && (7) \\
\sigma_z \otimes \sigma_x | \uparrow\downarrow\rangle &= | \uparrow\uparrow\rangle && (8) \\
\sigma_z \otimes \sigma_x | \downarrow\uparrow\rangle &= -| \downarrow\downarrow\rangle && (9) \\
\sigma_z \otimes \sigma_x | \downarrow\downarrow\rangle &= -| \downarrow\uparrow\rangle && (10) \\
& && (11)
\end{aligned}$$

and hence

$$\bar{M} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{12}$$

1.2. Consider an arbitrary state

$$|\psi\rangle = \psi_{\uparrow\uparrow}| \uparrow\uparrow\rangle + \psi_{\uparrow\downarrow}| \uparrow\downarrow\rangle + \psi_{\downarrow\uparrow}| \downarrow\uparrow\rangle + \psi_{\downarrow\downarrow}| \downarrow\downarrow\rangle, \tag{13}$$

where the coefficients $\psi_{\sigma\tau}$ are complex numbers. Imagine creating a new state

$$\begin{aligned}
|\psi'\rangle &= \sigma_z \otimes \sigma_x |\psi\rangle && (14) \\
&= \psi'_{\uparrow\uparrow}| \uparrow\uparrow\rangle + \psi'_{\uparrow\downarrow}| \uparrow\downarrow\rangle + \psi'_{\downarrow\uparrow}| \downarrow\uparrow\rangle + \psi'_{\downarrow\downarrow}| \downarrow\downarrow\rangle. && (15)
\end{aligned}$$

The coefficients of the new state are related to the old via

$$\begin{pmatrix} \psi'_{\uparrow\uparrow} \\ \psi'_{\uparrow\downarrow} \\ \psi'_{\downarrow\uparrow} \\ \psi'_{\downarrow\downarrow} \end{pmatrix} = \begin{pmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix} \begin{pmatrix} \psi_{\uparrow\uparrow} \\ \psi_{\uparrow\downarrow} \\ \psi_{\downarrow\uparrow} \\ \psi_{\downarrow\downarrow} \end{pmatrix}. \tag{16}$$

Call this matrix M. Find M.

Solution 1.2. (2 points) Applying the operator to the state yields

$$\begin{aligned}
\sigma_z \otimes \sigma_x |\psi\rangle &= \psi_{\uparrow\uparrow}\sigma_z \otimes \sigma_x |\uparrow\uparrow\rangle + \psi_{\uparrow\downarrow}\sigma_z \otimes \sigma_x |\uparrow\downarrow\rangle + \psi_{\downarrow\uparrow}\sigma_z \otimes \sigma_x |\downarrow\uparrow\rangle + \psi_{\downarrow\downarrow}\sigma_z \otimes \sigma_x |\downarrow\downarrow\rangle && (17)\\
&= \psi_{\uparrow\uparrow}|\uparrow\downarrow\rangle + \psi_{\uparrow\downarrow}|\uparrow\uparrow\rangle - \psi_{\downarrow\uparrow}|\downarrow\downarrow\rangle - \psi_{\downarrow\downarrow}|\downarrow\uparrow\rangle. && (18)
\end{aligned}$$

Equating terms yields

$$\begin{aligned}
\psi'_{\uparrow\uparrow} &= \psi_{\uparrow\downarrow} && (19)\\
\psi'_{\uparrow\downarrow} &= \psi_{\uparrow\uparrow} && (20)\\
\psi'_{\downarrow\uparrow} &= -\psi_{\downarrow\downarrow} && (21)\\
\psi'_{\downarrow\downarrow} &= -\psi_{\downarrow\uparrow} && (22)
\end{aligned}$$

and hence

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{23}$$

1.3. How is $\bar{M}$ related to M?

Solution 1.3. (1 point)
In this case $\bar{M} = M$.
As an aside, in general $\bar{M} = M^t$.
To see the general case, consider a general operator $\hat{X}$, and a general basis $|j\rangle$, where j indexes the states. The action of $\hat{X}$ on the basis states can be expressed as

$$\hat{X}|i\rangle = \sum_j (\bar{X})_{ij}|j\rangle, \tag{24}$$

which defines the matrix $\bar{X}$. Consider a general state $|\psi\rangle = \sum_j \psi_j |j\rangle$. Acting on it with $\hat{X}$ yields

$$\begin{aligned}
\hat{X}|\psi\rangle &= \sum_{jk} \psi_j (\bar{X})_{jk}|k\rangle && (25)\\
&= \sum_k \psi'_k |k\rangle. && (26)
\end{aligned}$$

1.4. A very important operator for two spin-1/2 objects is

$$\hat{\Lambda} = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z. \tag{27}$$

One reason it is important is that it is "rotationally invariant". That is, it under rotation of space it maps onto itself. You will spend quite some time on the rotational properties of operators in PHYS 4443.

Generate the 4×4 matrix defined by

$$\hat{\Lambda} \begin{pmatrix} |\uparrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{pmatrix} = \begin{pmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix} \begin{pmatrix} |\uparrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{pmatrix}. \tag{28}$$

Call the matrix $\bar{\Lambda}$. [Hint: It is easiest to separately do the three terms in Eq. (27), then add them up.]

Solution 1.4. (3 points)
Lets denote the three matrices as Λ_x, Λ_y, and Λ_z.

$$\bar{\Lambda}_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{29}$$

$$\bar{\Lambda}_y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{30}$$

$$\bar{\Lambda}_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{31}$$

and hence

$$\bar{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{32}$$

1.5. Find the eigenvalues of $\bar{\Lambda}$ (including multiplicities).

Solution 1.5. (1 point)
The eigenvalues are: 1,1,1,-3.
One context one sees this matrix in is when you look at the square of the total spin operator:

$$|\vec{S}|^2 = |\vec{S}_1 + \vec{S}_2|^2 = |\vec{S}_1|^2 + |\vec{S}_2|^2 + 2\vec{S}_1 \cdot \vec{S}_2 \tag{33}$$

The first two terms are each proportional to the identity (they each yield $3\hbar^2/4$. The last term is the matrix $(\hbar^2/2)\hat{\Lambda}$. Thus the eigenvalues of $|\vec{S}|^2$ are $2\hbar^2, 2\hbar^2, 2\hbar^2, 0$ – representing three states that transform as a spin-1 object, and one that transforms as a spin-0.

Problem 2. Symmetric Bell's Inequalities

In class we showed that quantum mechanics was incompatible with a local hidden variables theory. In particular, we imagined a setting where two spins were created in a completely correlated state. Each spin hits a detector. The detectors determine the polarization along one of three randomly chosen directions – each 120° from one another.

We showed that quantum mechanically, whenever the two detectors point in the same direction, the two spins will be detected to be aligned. [This is obvious – it is what we mean by completely correlated.] We argued that when the two detectors were pointing in different directions the spins will be "aligned" 25% of the time. By this I mean if one spin is "up" in the $\hat{z}$ direction, then the other will be up in the 120° basis 25% of the time. We will call this probability 25% the "coincidence probability".

Finally we argued that this was inconsistent with a hidden variable theory.

In a hidden variable theory, each spin has an identical codebook, which tells it if it should be up or down in the three directions. Here you will calculate from symmetry the probabilities of different "codebooks." In particular, let $P(\sigma_1, \sigma_2, \sigma_3)$ be the probability that the codebook has the spin being σ_1 in the $\hat{z}$ direction, σ_2 in the 120° direction, and σ_3 in the $-120°$ direction. [Note, the codebook is *not* probabilistic. This is the probability for a given codebook to be assigned to the spin.]

By symmetry

$$P(\uparrow,\uparrow,\uparrow) = P(\downarrow,\downarrow,\downarrow) = p \tag{34}$$

$$P(\uparrow,\uparrow,\downarrow) = P(\uparrow,\downarrow,\uparrow) = P(\downarrow,\uparrow,\uparrow) = P(\downarrow,\downarrow,\uparrow) = P(\downarrow,\uparrow,\downarrow) = P(\uparrow,\downarrow,\downarrow) = c. \tag{35}$$

2.1. Given that the sum of all probabilities is unity, write a relationship between p and c.

Solution 2.1. (1 point)
All the possibilites and their probabilities have been listed, so their probabilities should add to 1.

$$2p + 6c = 1 \tag{36}$$

2.2. Given that both p and c must be non-negative, what is the largest c can be?

Solution 2.2. (1 point)
Since $p \geq 0$, we have

$$c = \frac{1-2p}{6} \leq \frac{1}{6} \tag{37}$$

2.3. What is the smallest that the coincident probability can be:

$$P_{\text{coinc}} = P(\uparrow,\uparrow,\uparrow) + P(\uparrow,\uparrow,\downarrow) + P(\downarrow,\downarrow,\uparrow) + P(\downarrow,\downarrow,\downarrow) \tag{38}$$

This is one form of "Bell's Inequality" – and it is clearly violated by the quantum prediction (and the experiment).

Solution 2.3. (1 point)
Let us choose the first detector to be in the $\hat{z}$ direction and the second detector in the 120° direction. The coincidence probability is then given by

$$\begin{aligned} P_{\text{coinc}} &= P(\uparrow,\uparrow,\uparrow) + P(\uparrow,\uparrow,\downarrow) + P(\downarrow,\downarrow,\uparrow) + P(\downarrow,\downarrow,\downarrow) \\ &= 2p + 2c \\ &= 1 - 4c \\ &\geq \frac{1}{3} \end{aligned} \tag{39}$$

We see that the minimum coincidence probability is 33%, whereas quantum mechanics allows for a coincidence probability of 25% for a completely correlated state.

Problem 3. Entanglement A state of two spins is called a product state if it can be written as

$$|\psi\rangle = (a|\uparrow\rangle + b|\downarrow\rangle) \otimes (c|\uparrow\rangle + d|\downarrow\rangle). \tag{40}$$

For a product state, a measurement of one spin tells you nothing about the state of the other.

For contrast, a state is called entangled if it cannot be written as a product state. An example is the Bell state

$$|\psi\rangle = \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}. \tag{41}$$

3.1. Prove that the Bell state is entangled. Hint: try to find what a, b, c, d are, and see if you get a contradiction.

Solution 3.1. (2 points)
The product state given in the question can be written as

$$|\psi\rangle = ac|\uparrow\uparrow\rangle + bc|\downarrow\uparrow\rangle + ad|\uparrow\downarrow\rangle + bd|\downarrow\downarrow\rangle \tag{42}$$

A Bell state requires $ac = bd = \frac{1}{\sqrt{2}}$ and $ad = bc = 0$. It is clear that no solutions for a, b, c and d exist. For example, we find that $ac \times bd = abcd = \frac{1}{2}$ but $bc \times ad = abcd = 0$, which is a contradiction.

3.2. Find one other entangled state, and prove that it is entangled.

Solution 3.2. (2 points)
Another example of an entangled state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) \tag{43}$$

This state requires $ad = bc = \frac{1}{\sqrt{2}}$ and $ac = bd = 0$, which again leads to a contradiction. Other common choices are $|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle$, or $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$.

Problem 4. Shor's Algorithm

Here I will walk you through applying Shor's algorithm for factoring the number 187. This is a small enough number that you can run the algorithm by hand – with minimal help from a classical computer.

4.1. The first step is to pick a random a. Lets choose $a = 7$. We then define $f(x) = \mod(a^x, 187)$. Find the period of f. [I recommend using a computer. The Python command for mod is %. Rather than just making a loop of powers of 7, then applying the `mod` operator to them, I would generate subsequent terms from the last via: `f(x+1) = mod( a f(x),187)`. By this I mean you have a loop where you at each time through the loop you take $f \to \mod(af, 187)$, and $x \to x + 1$. You stop the loop as soon as $f = 1$. The x which corresponds to that f is the period.

A good way to check that your loop is working is to try it with a simpler case, such as $f(x) = \mod(3^x, 7)$, which you can do by hand: $f(0) = 1, f(1) = 3, f(2) = 2, f(3) = 6, f(4) = 4, f(5) = 5, f(6) = 1, f(7) = 3, \ldots$ This sequence has period 6 (which your program should be able to find).

If the loop stymies you, the other approach is to just generate a few hundred values of f, and look for the first repeat.

Solution 4.1. (2 points)
However you do it, you should find that the period is 80.

The role of quantum mechanics in Shor's algorithm is finding this period. The whole idea is that one can use quantum interference to find peaks in the discrete Fourier transform of this sequence, which lets you deduce the period.

4.2. If r is the period, and r is even, then Number Theory tells us that $f(r/2) \pm 1$ will have common factors with 187. Use this result to find the factors of 187. Here is a quick function which finds the greatest common divisor of two numbers:

```
def gcd(a,b):
     while b:
          a,b=b,a%b
     return a
```

This algorithm was known to the Greeks, and is called "Euclid's Algorithm."

Solution 4.2. (2 points)
You should find that for $r = 80$, $f(r/2) - 1 = 66$ and $f(r/2) + 1 = 68$. 66 and 187 have common factors 11, while 68 and 187 have common factors 17. Since $187 = 11 \times 17$, we have indeed factored 187.

Problem 5.

5.1. What are the values of s such that

$$e^{is} = 1? \tag{44}$$

Hint: There are infinitely many of them.

Solution 5.1. (1 point) $s = 2n\pi$ for all $n \in \mathbb{Z}$.

5.2. For given θ, what are the values of s such that

$$e^{is} = e^{i\theta}. \tag{45}$$

Hint: Question 5.1 is the special case $\theta = 0$, so there should be infinitely many solutions.

Solution 5.2. (1 point) $s = 2n\pi + \theta$ for all $n \in \mathbb{Z}$.

5.3. For fixed real A and ϕ, what are the values of s and t such that

$$e^{is+t} = Ae^{i\phi}? \tag{46}$$

Solution 5.3. Assuming that A is positive: $s = 2n\pi + \phi$ for all $n \in \mathbb{Z}$, and $t = \ln(A)$.

One often summarizes these results by defining the natural log to be a multi-valued function,

$$\ln(z) = \ln(|z|) + i\arg(z) + 2\pi i n. \tag{47}$$

Your calculator typically gives this quantity with $n = 0$, called the "principal branch." Similar, arcsin and arccos, defined as the inverse of sin and cos are most naturally thought of as multivalued functions, and your calculator only gives the principal branch.

Problem 6. Consider the function $F = z^* z + (2+i)z + (2-i)z^* + 3$, where $z = x + iy$ and $z^* = x - iy$.

6.1. Write F in terms of x and y.

Solution 6.1. (1 point)

$$F = x^2 + y^2 + 4x - 2y + 3. \tag{48}$$

6.2. Minimize this function by evaluating

$$\left(\frac{\partial F}{\partial x}\right)_y = 0 \tag{49}$$

$$\left(\frac{\partial F}{\partial y}\right)_x = 0 \tag{50}$$

where in the first line one takes the derivative with respect to x for fixed y, and in the second line one takes the derivative with respect to y for fixed x.

Solution 6.2. (2 points) Taking the derivatives, we find

$$2x + 4 = 0 \tag{51}$$

$$2y - 2 = 0, \tag{52}$$

and hence $x = -2$, $y = 1$.

6.3. Suppose we formally treat z and z^* as independent variables. One can then look at the equation

$$\left(\frac{\partial F}{\partial z^*}\right)_z = 0. \tag{53}$$

You should find that this is an equation for z. Solve it. How is this z related to the result of Problem 6.2?

Solution 6.3. (1 points) Taking the derivative, we find

$$z + (2 - i) = 0 \tag{54}$$

which gives hence $x = -2$, $y = 1$, as before.

Problem 7. Feedback

7.1. How long did this homework take?

7.2. Which of the following words come to mind when you think about this homework (feel free to add your own words if you have something better): frustrating, fun, tedious, insightful, hard, easy, useful, useless, fair, unfair