P3317 HW from Lecture 12+13 and Recitation 7

Due Oct 16, 2018

Problem 1. Separation of variables Suppose we have two masses that can move in 1D. They are attached by a spring, yielding a Hamiltonian

$$H = -\frac{\hbar^2}{2m}\partial_1^2 - \frac{\hbar^2}{2m}\partial_2^2 + \frac{1}{2}k(x_1 - x_2)^2, \tag{1}$$

where the derivative symbols mean

$$\partial_1\psi = \left(\frac{\partial\psi}{\partial x_1}\right)_{x_2} \tag{2}$$

$$\partial_2\psi = \left(\frac{\partial\psi}{\partial x_2}\right)_{x_1}. \tag{3}$$

We will attempt to write the two-particle wavefunction $\psi(x_1, x_2)$ in terms of

$$X = \frac{x_1 + x_2}{2} \tag{4}$$

$$x = (x_1 - x_2). \tag{5}$$

1.1. Using your knowledge of multivariable calculus, write

$$\partial_{x_1}\psi = \left(\frac{\partial\psi}{\partial x_1}\right)_{x_2} \tag{6}$$

in terms of

$$\partial_X\psi = \left(\frac{\partial\psi}{\partial X}\right)_x \tag{7}$$

and

$$\partial_x\psi = \left(\frac{\partial\psi}{\partial x}\right)_X. \tag{8}$$

Equations (7) and (8) define the operators ∂_X and ∂_x.

Solution 1.1 (2 points)**.** Using chain rule

$$\left(\frac{\partial\psi}{\partial x_1}\right)_{x_2} = \left(\frac{\partial\psi}{\partial X}\right)_x \left(\frac{\partial X}{\partial x_1}\right)_{x_2} + \left(\frac{\partial\psi}{\partial x}\right)_X \left(\frac{\partial x}{\partial x_1}\right)_{x_2} \tag{9}$$

$$= \frac{1}{2}\left(\frac{\partial\psi}{\partial X}\right)_x + \left(\frac{\partial\psi}{\partial x}\right)_X \tag{10}$$

Thus we can write

$$\partial_{x_1} = \frac{1}{2}\partial_X + \partial_x \tag{11}$$

1.2. Do the same with

$$\partial_2 \psi = \left(\frac{\partial \psi}{\partial x_2} \right)_{x_1} \tag{12}$$

Solution 1.2 (2 points)**.** Using chain rule

$$\begin{aligned} \left(\frac{\partial \psi}{\partial x_2} \right)_{x_1} &= \left(\frac{\partial \psi}{\partial X} \right)_x \left(\frac{\partial X}{\partial x_2} \right)_{x_1} + \left(\frac{\partial \psi}{\partial x} \right)_X \left(\frac{\partial x}{\partial x_2} \right)_{x_1} & (13) \\ &= \left(\frac{\partial \psi}{\partial X} \right)_x \frac{1}{2} - \left(\frac{\partial \psi}{\partial x} \right)_X & (14) \end{aligned}$$

Thus we can write

$$\partial_{x_2} = \frac{1}{2}\partial_X - \partial_x \tag{15}$$

1.3. Write

$$\hat{T} = -\frac{\hbar^2}{2m}\partial_1^2 - \frac{\hbar^2}{2m}\partial_2^2 \tag{16}$$

in terms of derivatives with respect to X and x.

Solution 1.3 (2 points)**.**

$$\begin{aligned} \partial_1^2 + \partial_2^2 &= \left(\frac{1}{2}\partial_X + \partial_x \right)^2 + \left(\frac{1}{2}\partial_X - \partial_x \right)^2 & (17) \\ &= \frac{1}{2}\partial_X^2 + 2\partial_x^2 & (18) \end{aligned}$$

Thus

$$\hat{T} = -\frac{\hbar^2}{4m}\partial_X^2 - \frac{\hbar^2}{m}\partial_x^2 \tag{19}$$

1.4. Show that the time independent Schrodinger equation

$$\hat{H}\psi = E\psi, \tag{20}$$

can be satisfied by a wavefunction of the form

$$\psi = f(X)g(x), \tag{21}$$

where f only depends on the center-of-mass coordinate X, and g only depends on the relative co-ordinate x. Find the equations which must be satisfied by f and g.

Solution 1.4 (3 points)**.** We begin by assuming that the solution has that form, and hence

$$E f(X) g(x) \quad = \quad g(x)\left[-\frac{\hbar^2}{4m} f''(X)\right] + f(x)\left[-\frac{\hbar^2}{m} g''(x) + \frac{1}{2} m\omega^2 x^2 g(x)\right]. \tag{22}$$

Dividing this expression by $f(X)g(x)$ yields

$$E = \frac{1}{f(X)}\left[-\frac{\hbar^2}{4m} f''(X)\right] + \frac{1}{g(x)}\left[-\frac{\hbar^2}{m} g''(x) + \frac{1}{2} m\omega^2 x^2 g(x)\right]. \tag{23}$$

This can only be satisfied if there are two numbers E_1 and E_2 such that

$$E_1 f(X) \quad = \quad -\frac{\hbar^2}{4m} f''(X) \tag{24}$$

$$E_2 g(x) \quad = \quad -\frac{\hbar^2}{m} g''(x) + \frac{1}{2} m\omega^2 x^2 g(x). \tag{25}$$

We then have $E = E_1 + E_2$.

Problem 2. Ladder operators This is a review of material covered in PHYS 3316.

Using our standard tricks, we adimensionalize the Hamiltonian for the simple harmonic oscillator, writing

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{x}^2, \tag{26}$$

where the dimensionless position and momentum operators obey

$$[\hat{x}, \hat{p}] = i. \tag{27}$$

[To get back to dimensional quantities, you would multiply energies by $\hbar\omega$, lengths by $\sqrt{\hbar/m\omega}$ and momenta by $\sqrt{\hbar m\omega}$. For this question, however, use the dimensionless quantities.]

2.1. Consider the operators $\hat{a} = (\hat{x} + i\hat{p})/\sqrt{2}$ and $\hat{a}^\dagger = (\hat{x} - i\hat{p})/\sqrt{2}$. Find the commutator $[\hat{a}, \hat{a}^\dagger]$.

Solution 2.1 (2 points)**.**

$$[\hat{a}, \hat{a}^\dagger] \quad = \quad \frac{1}{2}[\hat{x} + i\hat{p}, \hat{x} - i\hat{p}] \tag{28}$$

$$= \quad \frac{1}{2}\left([\hat{x}, \hat{x}] + i[\hat{p}, \hat{x}] - i[\hat{x}, \hat{p}] + [\hat{p}, \hat{p}]\right) \tag{29}$$

$$= \quad 1 \tag{30}$$

2.2. Show that the Hamiltonian can be written as

$$\hat{H} = \alpha\, \hat{a}^\dagger \hat{a} + \beta. \tag{31}$$

Find the numbers α and β.

Solution 2.2 (2 points)**.** We first note

$$\begin{aligned}\hat{a}^\dagger\hat{a} &= \frac{1}{2}(\hat{x}-i\hat{p})(\hat{x}+i\hat{p}) && (32)\\ &= \frac{1}{2}\left(\hat{x}^2+i(\hat{x}\hat{p}-\hat{p}\hat{x})+\hat{p}^2\right) && (33)\\ &= \frac{1}{2}\left(\hat{x}^2+\hat{p}^2\right)-\frac{1}{2}, && (34)\end{aligned}$$

which then gives

$$\hat{H} = \hat{a}^\dagger\hat{a}+\frac{1}{2}, \tag{35}$$

and hence $\alpha = 1, \beta = 1/2$.

2.3. Calculate $[\hat{a},\hat{H}]$ and $[\hat{a}^\dagger,\hat{H}]$. Hint: use $[A,BC] = B[A,C]+[A,B]C$.

Solution 2.3 (2 points)**.**

$$\begin{aligned}[\hat{a},\hat{H}] &= [\hat{a},\hat{a}^\dagger\hat{a}] && (36)\\ &= [\hat{a},\hat{a}^\dagger]\hat{a} && (37)\\ &= \hat{a}. && (38)\end{aligned}$$

Similarly

$$\begin{aligned}[\hat{a}^\dagger,\hat{H}] &= [\hat{a}^\dagger,\hat{a}^\dagger\hat{a}] && (39)\\ &= \hat{a}^\dagger[\hat{a}^\dagger,\hat{a}] && (40)\\ &= -\hat{a}^\dagger. && (41)\end{aligned}$$

2.4. Suppose $|\psi\rangle$ is an eigenstate of $\hat{H}$ with dimensionless energy E. Show that $|\psi'\rangle = \hat{a}|\psi\rangle$ is also an eigenstate. How is its energy, E' related to E? [Hint: Use the commutation relationship between $\hat{a}$ and $\hat{H}$.]

Solution 2.4 (2 points)**.**

$$\begin{aligned}\hat{H}|\psi'\rangle &= \hat{H}\hat{a}|\psi\rangle && (42)\\ &= \left(\hat{H}\hat{a}-\hat{a}\hat{H}+\hat{a}\hat{H}\right)|\psi\rangle && (43)\\ &= \left(-\hat{a}+\hat{a}\hat{H}\right)|\psi\rangle && (44)\\ &= (E-1)|\psi'\rangle. && (45)\end{aligned}$$

Thus $|\psi'\rangle$ is an energy eigenstate with eigenvalue $E-1$.

2.5. Suppose $|\psi\rangle$ is an eigenstate of $\hat{H}$ with dimensionless energy E. Show that $|\psi'\rangle = \hat{a}^\dagger|\psi\rangle$ is also an eigenstate. How is its energy, E' related to E? [Hint: Use the commutation relationship between $\hat{a}^\dagger$ and $\hat{H}$.]

Solution 2.5 (2 points).

$$\begin{aligned}
\hat{H}|\psi'\rangle &= \hat{H}\hat{a}^\dagger|\psi\rangle && (46)\\
&= \left(\hat{H}\hat{a}^\dagger - \hat{a}^\dagger\hat{H} + \hat{a}^\dagger\hat{H}\right)|\psi\rangle && (47)\\
&= \left(\hat{a}^\dagger + \hat{a}^\dagger\hat{H}\right)|\psi\rangle && (48)\\
&= (E+1)|\psi'\rangle. && (49)
\end{aligned}$$

Thus $|\psi'\rangle$ is an energy eigenstate with eigenvalue $E+1$.

2.6. In PHYS 3316 you should have further argued that there is a ground state $|0\rangle$ defined by $\hat{a}|0\rangle = 0$. Find its energy, E_0.

2.7 (2 points).

$$\begin{aligned}
\hat{H}|0\rangle &= \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|0\rangle && (50)\\
&= \frac{1}{2}|0\rangle. && (51)
\end{aligned}$$

Thus $E_0 = 1/2$.

In PHYS 3316 you also should have calculated the normalization of the states. I will not have you work through it here, but the result that you found was that

$$\begin{aligned}
\hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle && (52)\\
\hat{a}|n\rangle &= \sqrt{n}|n-1\rangle && (53)
\end{aligned}$$

Problem 3. Degeneracies of the 2D Harmonic Oscillator

3.1. The eigenstates of the 2D harmonic oscillator can be labeled by 2 quantum numbers, $n_x, n_y = 0, 1, \cdots$ corresponding to the number of quanta of excitation in each direction. What is the energy of the state with quantum numbers n_x, n_y? Take the spring constant to be equal in both direction, and take the classical oscillation frequency to be ω.

Solution 3.1 (1 point).

$$E = \hbar\omega\,(1/2 + n_x) + \hbar\omega\,(1/2 + n_y) = \hbar\omega\,(1 + n_x + n_y)\,, \qquad (54)$$

3.2. Make a table that has three columns. In the first column put the energy E. In the second column list all of the n_x, n_y combinations which make that energy. In the third column put the total degeneracy of the energy level. Fill in the first four energies.

Solution 3.2 (3 points)**.**

$E/(\hbar\omega)$	(n_x, n_y)	Degeneracy
1	$(0,0)$	1
2	$(1,0),(0,1)$	2
3	$(2,0),(1,1),(0,2)$	3
4	$(3,0),(2,1),(1,2),(0,3)$	4

(55)

3.3. What is the degeneracy of the n'th level? Note: this degeneracy is in many ways analogous to the large degeneracy in the Hydrogen spectrum. In both cases all classical trajectories are closed. Here we have the additional features that all classical orbits have the same period. As emphasized in class, degeneracies are almost always related to symmetries.

Solution 3.3 (2 points)**.** Starting counting at $n = 1$ (so $E = n\hbar\omega$) the degeneracy will be n. If you start counting at $n = 0$, the degeneracy will be $n + 1$. Either answer is fine.

Julien Schwinger, one of the great physicists of the 20th century, came up with an ingenious way to map the symmetry in this problem onto something which looks like angular momentum. I should emphasize however that it is not angular momentum. He started with the ladder operators for the x and y oscillators:

$$a_x|n_x, n_y\rangle = \sqrt{n_x}|n_x - 1, n_y\rangle \tag{56}$$

$$a_y|n_x, n_y\rangle = \sqrt{n_y}|n_x, n_y - 1\rangle. \tag{57}$$

He then created the following combinations:

$$\mathcal{L}_x = \frac{a_x^\dagger a_y + a_y^\dagger a_x}{2} \tag{58}$$

$$\mathcal{L}_y = \frac{a_x^\dagger a_y - a_y^\dagger a_x}{2i} \tag{59}$$

$$\mathcal{L}_z = \frac{a_x^\dagger a_x - a_y^\dagger a_y}{2}. \tag{60}$$

The first two are traditionally combined to make

$$\mathcal{L}_+ = a_x^\dagger a_y \tag{61}$$

$$\mathcal{L}_- = a_y^\dagger a_x. \tag{62}$$

3.4. I would like you to prove that $\mathcal{L}$ obeys the same commutation relations as angular momentum:

$$[\mathcal{L}_\pm, \mathcal{L}_z] = \mp\mathcal{L}_\pm \tag{63}$$

$$[\mathcal{L}_+, \mathcal{L}_-] = 2\mathcal{L}_z \tag{64}$$

Tricks:
Use the identity

$$[AB, CD] = AC[B, D] + A[B, C]D + C[A, D]B + [A, C]BD,$$

and the relations

$$\begin{aligned}
[a_x, a_y] &= [a_x, a_y^\dagger] = [a_x^\dagger, a_y] = [a_x^\dagger, a_y^\dagger] = 0 \\
[a_x, a_x^\dagger] &= [a_y, a_y^\dagger] = 1 \\
[a_x, a_x] &= [a_x^\dagger, a_x^\dagger] = [a_y, a_y] = [a_y^\dagger, a_y^\dagger] = 0
\end{aligned}$$

Using these tricks, the identities should not take more than a few lines to prove. If you get stuck, skip it. I will give you full marks if you write "I spent 30 minutes on this part of the problem, and could not complete it." [Don't forget to complete the other questions though – particularly the one at the end where you look at the anisotropic oscillator.]

Solution 3.4 (3 points)**.** Starting with:

$$[\mathcal{L}_+, \mathcal{L}_z] = \frac{1}{2}\left([a_x^\dagger a_y, a_x^\dagger a_x] - [a_x^\dagger a_y, a_y^\dagger a_y]\right), \tag{65}$$

we use the identities and pick out only those terms that are non-zero:

$$\begin{aligned}
[\mathcal{L}_+, \mathcal{L}_z] &= \frac{1}{2}\left(a_x^\dagger[a_x^\dagger, a_x]a_y - a_x^\dagger[a_y, a_y^\dagger]a_y\right), &(66)\\
&= -a_x^\dagger a_y = -\mathcal{L}_+. &(67)
\end{aligned}$$

Similarly:

$$\begin{aligned}
[\mathcal{L}_+, \mathcal{L}_z] &= \frac{1}{2}\left([a_y^\dagger a_x, a_x^\dagger a_x] - [a_y^\dagger a_x, a_y^\dagger a_y]\right), &(68)\\
&= \frac{1}{2}\left(a_y^\dagger[a_x, a_x^\dagger]a_x - a_y^\dagger[a_y^\dagger, a_y]a_x\right), &(69)\\
&= -a_x^\dagger a_y = +\mathcal{L}_-. &(70)
\end{aligned}$$

Finally,

$$\begin{aligned}
[\mathcal{L}_+, \mathcal{L}_-] &= [a_x^\dagger a_y, a_y^\dagger a_x] &(71)\\
&= \frac{1}{2}\left(a_y^\dagger[a_x^\dagger, a_x]a_y + a_x^\dagger[a_y, a_y^\dagger]a_x\right), &(72)\\
&= -a_y^\dagger a_y + a_x^\dagger a_x = 2\mathcal{L}_z. &(73)
\end{aligned}$$

Of course, you will get full marks if you write "I spent 30 minutes on this part of the problem, and could not complete it."

3.5. Prove that $\mathcal{L}_\pm$ and $\mathcal{L}_z$ commute with the Hamiltonian – and hence are generators of a symmetry.

Again, using the tricks in 3.4, this should only take a couple lines. Again, don't spend more than another 30 minutes on this one. You will get full marks if you write "I spent 30 minutes on this part of the problem, and could not complete it." [Don't forget to complete the other questions though – particularly the one at the end where you look at the anisotropic oscillator.]

Solution 3.5 (3 points)**.** The Hamiltonian is given by:

$$H = \hbar\omega(1 + a_x^\dagger a_x + a_y^\dagger a_y). \tag{74}$$

Therefore:

$$\begin{aligned} [\mathcal{L}_+, H] &= \hbar\omega\left([a_x^\dagger a_y, a_x^\dagger a_x] + [a_x^\dagger a_y, a_y^\dagger a_y]\right) && (75)\\ &= \hbar\omega\left(a_x^\dagger [a_x^\dagger, a_x] a_y + a_x^\dagger [a_y, a_y^\dagger] a_y\right) && (76)\\ &= 0 && (77)\end{aligned}$$

and similarly for $\mathcal{L}_-$. For $\mathcal{L}_z$:

$$\begin{aligned} [\mathcal{L}_z, H] &= \frac{\hbar\omega}{2}\left([a_x^\dagger a_x, a_x^\dagger a_x] + [a_y^\dagger a_y, a_y^\dagger a_y]\right) && (78)\\ &= \frac{\hbar\omega}{2}\left([n_x, n_x] + [n_y, n_y]\right) && (79)\\ &= 0, && (80)\end{aligned}$$

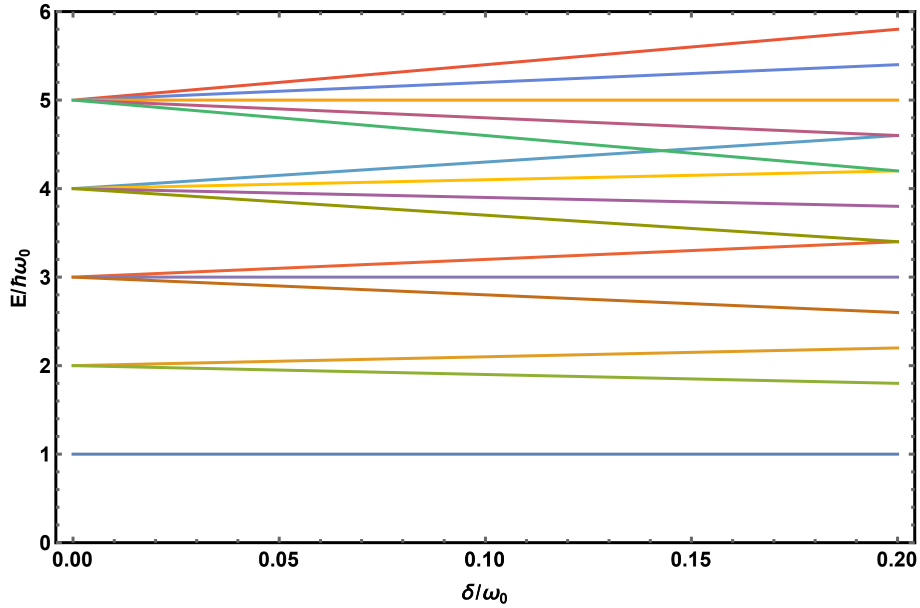
where, taking a break from repeated use of commutation identities, we have simply used our physical intuition/knowledge that number operators commute with themselves.
Of course, you will get full marks if you write "I spent 30 minutes on this part of the problem, and could not complete it."

[As an aside – the rotations form a group structure. The group is often labeled SU(2) as it is locally isomorphic to the 2×2 unitary, unimodular, matrices. One can do a similar trick of finding representations of $SU(3)$ by looking at the the states of a 3-dimensional harmonic oscillator. $SU(3)$ is important as it is one of the symmetries of the standard model of particle physics. The degeneracies of the 3D harmonic oscillator are: $1, 3, 6, 10, 15...$ and this construction explicitly produces representations of $SU(3)$ with those dimension. It turns out that this trick does not exhaustively enumerate the representations of SU(3). For example, when we talk about Mesons, we will see that there are 8-dimensional representations of SU(3).]

3.6. Suppose we have a slightly anisotropic oscillator: $\omega_x = \omega_0 + \delta$, $\omega_y = \omega_0 - \delta$. The frequency δ quantifies the breaking of the symmetry between x and y. In particular, it means that classical trajectories are no longer closed.

Make a plot of $E/\hbar\omega_0$ vs δ/ω_0 for the 15 lowest energy states. Go from $\delta = 0$ to $\delta = 0.2\omega_0$. Properly label your axes. Describe qualitatively how the energy levels respond to breaking the symmetry.

Solution 3.6 (5 points)**.** The anisotropy completely breaks the degeneracies of the 2D harmonic oscillator, splitting each originally degenerate multiplet into sets of states with evenly spaced energies proportional to δ.



Problem 4. Atomic physicists can excite atoms into very high energy orbits. These highly excited atoms are relatively stable, and are known as "Rydberg atoms." Here you will estimate their size.

4.1. What is the degeneracy of the $n = 137$ level in Hydrogen? Show your reasoning.

Solution 4.1 (4 points)**.** In the n'th energy level, there are states with orbital angular momentum quantum number l running over the integers from 0 to $n - 1$. For each l there are $2l + 1$ states $m = -l, -l + 1, ..., +l$. Therefore, the total number of orbital states is:

$$\sum_{l=0}^{n-1}(2l+1) = \sum_{l=1}^{n}(2l-1) = n(n+1) - n = n^2. \tag{81}$$

On top of this there are two electron spin states, meaning that the degeneracy is given by:

$$\text{Degeneracy} = 2n^2 = 37538 \tag{82}$$

4.2. What is the energy of a Hydrogen in the $n = 137$ level (in eV)? Take your origin of energy so that ionized Hydrogen has energy $E = 0$.

Solution 4.2 (2 points)**.** The energy of the n'th level of a Hydrogen atom is given by

$$E_n = -\frac{1 \text{ Ry}}{n^2}, \tag{83}$$

where 1 Ry = 13.6 eV is the ionization energy of Hydrogen. Therefore, for $n = 137$,

$$E_{137} = -7.2 \times 10^{-4} \text{ eV}. \tag{84}$$

4.3. High energy states like this are quite "classical," meaning one can understand them by classical reasoning. Consider a classical particle of mass m_e executing a circular orbit in an energy potential $V(r) = -e^2/(4\pi\epsilon_0 r)$. What is the energy of such a classical particle with radius r. Choose the origin of energy so that a stationary electron at $r = \infty$ would have zero energy. [Don't forget to include the kinetic energy from the orbiting.]

Solution 4.3 (3 points)**.** The easiest approach is to use the Virial theorem, KE + PE = PE/2. Let's first prove that theorem:

$$\partial_r V(r) = F_{\text{centrip}} = \frac{mv^2}{r} = \frac{2\text{KE}}{r} \tag{85}$$

$$\implies \text{KE} = \frac{r}{2}\partial_r V(r) = \frac{e^2}{8\pi\epsilon_0 r} \tag{86}$$

$$= -\frac{\text{PE}}{2}. \tag{87}$$

Therefore

$$E = -\frac{e^2}{8\pi\epsilon_0 r} \tag{88}$$

4.4. By comparing these expressions, estimate the size (in Angstroms) of a Rydberg atom in the $n = 137$ level.

Solution 4.4 (2 points)**.** Combining Eqs. (84) with (88), we get

$$r = \frac{1}{2}\frac{e^2}{4\pi\epsilon_0\hbar c}\frac{1}{7.2\times10^{-4}}\frac{\hbar c}{\text{eV}} \tag{89}$$

$$= \frac{1}{2}\frac{1}{137}\frac{1}{7.2\times10^{-4}}\times 1.97\times10^{-7}\text{ m} \tag{90}$$

$$= 10^{-6}\text{ m} = 10^4\text{ Å}. \tag{91}$$

4.5. This classical reasoning becomes less accurate for smaller values of n, but since it is dimensionally correct, it gives the right order of magnitude. What size (in Angstroms) do you get if you repeat with $n = 1$?

Solution 4.5 (2 points)**.** With $n = 1$ we get:

$$r = \frac{1}{2}\frac{1}{137}\frac{1}{13.6}\times 1.97\times10^{-7}\text{ m} \tag{92}$$

$$= 0.5\text{ Å}, \tag{93}$$

which is surprisingly good!

Problem 5. The following wavefunction describes two particles in one-dimension

$$\psi(x_1, x_2) = A\left[\exp(-(x_1-a)^2/d^2 - (x_2+a)^2/d^2) - \exp(-(x_2-a)^2/d^2 - (x_1+a)^2/d^2)\right], \tag{94}$$

where A is a normalization constant, and a, d are lengths

5.1. What are the dimensions of A?

Solution 5.1 (2 points)**.** Since the wavefunction is normalised as

$$1 = \int dx_1 dx_2 |\psi|^2 = \int dx_1 dx_2 |A|^2 \left[e^{-\frac{(x_1-a)^2}{d^2} - \frac{(x_2+a)^2}{d^2}} - e^{-\frac{(x_2-a)^2}{d^2} - \frac{(x_1+a)^2}{d^2}}\right]^2, \tag{95}$$

we expect A to have dimension of $\frac{1}{\text{length}}$.

5.2. Can this wavefunction describe two identical particles? If so, would their statistics be fermionic or bosonic?

Solution 5.2 (2 points)**.** Let's see what happens when we swap x_1 and x_2

$$\begin{aligned}\psi &= A\left(e^{-\frac{(x_1-a)^2}{d^2}-\frac{(x_2+a)^2}{d^2}} - e^{-\frac{(x_2-a)^2}{d^2}-\frac{(x_1+a)^2}{d^2}}\right)\\ &\to A\left(e^{-\frac{(x_2-a)^2}{d^2}-\frac{(x_1+a)^2}{d^2}} - e^{-\frac{(x_1-a)^2}{d^2}-\frac{(x_2+a)^2}{d^2}}\right)\\ &= -\psi\end{aligned} \tag{96}$$

Therefore, this wavefunction can describe two identical particles with fermionic statistics.

5.3. What is the probability that particle one is in an interval of width dx, which is at the origin ($x = 0$)? You do not need to calculate A, and can include it in your answer.

Solution 5.3 (3 points)**.** Since we don't care about where particle two is, this is given by

$$\begin{aligned}P &= dx\int_{-\infty}^{\infty} dx_2|\psi(x_1=0,x_2)|^2\\ &= dx\int_{-\infty}^{\infty} dx_2|A|^2\left[e^{-\frac{(0-a)^2}{d^2}-\frac{(x_2+a)^2}{d^2}} - e^{-\frac{(x_2-a)^2}{d^2}-\frac{(0+a)^2}{d^2}}\right]^2\\ &= dx|A|^2e^{-\frac{2a^2}{d^2}}\int_{-\infty}^{\infty} dx_2\left[e^{-\frac{(x_2+a)^2}{d^2}} - e^{-\frac{(x_2-a)^2}{d^2}}\right]^2\\ &= dx|A|^2e^{-\frac{2a^2}{d^2}}\int_{-\infty}^{\infty} dx_2\left(e^{-\frac{2(x_2+a)^2}{d^2}} + e^{-\frac{2(x_2-a)^2}{d^2}} - 2e^{-\frac{2x_2^2+2a^2}{d^2}}\right)\\ &= dx|A|^2e^{-\frac{2a^2}{d^2}}\left(1+1-2e^{-\frac{2a^2}{d^2}}\right)\sqrt{\frac{\pi}{2}}d\\ &= dx|A|^2e^{-\frac{3a^2}{d^2}}\sinh\left(\frac{a^2}{d^2}\right)2\sqrt{2\pi}d\end{aligned} \tag{97}$$

The first line receives 2 points. The third point comes from doing the integral.

5.4. What is the probability that both particles are in and interval of width dx, which is at the origin?

Solution 5.4 (2 points)**.** This is given by

$$dxdx|\psi(x_1=0,x_2=0)|^2 = dx^2|A|^2\left[e^{-\frac{(0-a)^2}{d^2}-\frac{(0+a)^2}{d^2}} - e^{-\frac{(0-a)^2}{d^2}-\frac{(0+a)^2}{d^2}}\right]^2 = 0 \tag{98}$$

This is not surprising since the two fermions don't like to occupy the same position.

Problem 6. Given two orthonormal single-particle states $\phi_1(x)$ and $\phi_2(x)$, we can create a two-fermion state

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}} \left(\phi_1(x_1)\phi_2(x_2) - \phi_1(x_2)\phi_2(x_1)\right). \tag{99}$$

6.1. Can we make a three particle fermion state out of these two wavefunctions? Either construct it, or explain why it is impossible.

Solution 6.1 (2 points)**.** The answer is no. If we have three particles in two states, at least two of the particles need to be in the same state. There is no way to make an antisymmetric wavefunction with two particles in the same state.

6.2. Construct the analogous three-particle state made out of single particle states $\phi_1(x)$, $\phi_2(x)$, and $\phi_3(x)$.

Solution 6.2 (2 points)**.**

$$\begin{aligned}\psi(x_1, x_2, x_3) = \frac{1}{\sqrt{6}}\{&\phi_1(x_1)\phi_2(x_2)\phi_3(x_3) + \phi_2(x_1)\phi_3(x_2)\phi_1(x_3) + \phi_3(x_1)\phi_1(x_2)\phi_2(x_3) \\ &- \phi_1(x_1)\phi_3(x_2)\phi_2(x_3) - \phi_3(x_1)\phi_2(x_2)\phi_1(x_3) - \phi_2(x_1)\phi_1(x_2)\phi_3(x_3)\}\end{aligned} \tag{100}$$

6.3. If you have two non-interacting particles, the Hamiltonian will be

$$H = H_1 + H_2, \tag{101}$$

where the H_j are all the same, but the act on different particles. For example

$$H_1 = -\frac{\hbar^2}{2m}\partial_{x_1}^2 + V(x_1) \tag{102}$$

$$H_2 = -\frac{\hbar^2}{2m}\partial_{x_2}^2 + V(x_2). \tag{103}$$

Suppose ϕ_1 and ϕ_2 are eigenstates of the single particle Schrodinger equation,

$$H_1\phi_1(x_1) = \epsilon_1\phi_1(x_1) \tag{104}$$

$$H_2\phi_2(x_2) = \epsilon_2\phi_2(x_2). \tag{105}$$

What then is the energy of the state in Eq. (99).

Solution 6.3 (2 points)**.**

$$\begin{aligned}
H\psi &= (H_1 + H_2)\{\frac{1}{\sqrt{2}}(\phi_1(x_1)\phi_2(x_2) - \phi_1(x_2)\phi_2(x_1))\} \\
&= \frac{1}{\sqrt{2}}\{(H_1 + H_2)(\phi_1(x_1)\phi_2(x_2)) - (H_1 + H_2)(\phi_1(x_2)\phi_2(x_1))\} \\
&= \frac{1}{\sqrt{2}}\{(H_1\phi_1(x_1))\phi_2(x_2) + \phi_1(x_1)(H_2\phi_2(x_2)) - \phi_1(x_2)(H_1\phi_2(x_1)) - (H_2\phi_1(x_2))\phi_2(x_1)\} \\
&= \frac{1}{\sqrt{2}}\{\epsilon_1\phi_1(x_1)\phi_2(x_2) + \epsilon_2\phi_1(x_1)\phi_2(x_2) - \epsilon_2\phi_1(x_2)\phi_2(x_1) - \epsilon_1\phi_1(x_2)\phi_2(x_1)\} \\
&= (\epsilon_1 + \epsilon_2)\frac{1}{\sqrt{2}}(\phi_1(x_1)\phi_2(x_2) - \phi_1(x_2)\phi_2(x_1)) \\
&= (\epsilon_1 + \epsilon_2)\psi
\end{aligned} \tag{106}$$

The state has energy $\epsilon_1 + \epsilon_2$.

Problem 7. Feedback

7.1. How long did this homework take?

7.2. Which of the following words come to mind when you think about this homework (feel free to add your own words if you have something better): frustrating, fun, tedious, insightful, hard, easy, useful, useless, fair, unfair