# P3317 HW from Lecture 20+21 and Recitation 11

**Due Nov 13, 2018**

**Problem 1. Parameters in our model of sound**
As you know from the in-class exercise in lecture, this exercise is somewhat tedious, and requires careful book-keeping.

In class we modeled optical phonons via a mass-and-spring model

$$H = \sum_j \frac{p_j^2}{2m} + \frac{\kappa}{2} x_j^2 + \frac{\gamma}{2}(x_j - x_{j-1})^2. \tag{1}$$

We defined harmonic oscillator creation and annihilation operators,

$$x_j = \frac{d}{\sqrt{2}} \left( [(a_j + a_j^\dagger) + \alpha(a_{j+1} + a_{j+1}^\dagger + a_{j-1} + a_{j-1}^\dagger)\right] \tag{2}$$

$$p_j = \frac{\hbar}{\sqrt{2}di} \left( [(a_j - a_j^\dagger) - \alpha(a_{j+1} - a_{j+1}^\dagger + a_{j-1} - a_{j-1}^\dagger)\right]. \tag{3}$$

In the limit $\gamma \ll k$ (and $\alpha \ll 1$) we arrived at

$$\begin{aligned} H &= \sum_j \Big[ A(a_j^\dagger a_j + a_j a_j^\dagger) + B(a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}) \\ &\qquad + C(a_j a_j + a_j^\dagger a_j^\dagger) + D(a_j a_{j+1} + a_{j+1}^\dagger a_j^\dagger) + \cdots \Big]. \end{aligned} \tag{4}$$

I claimed that we can choose $\alpha$ and $d$ to make $C = D = 0$.

**1.1.** Choose *one* of the following three energies

$$K = \sum_j \frac{p_j^2}{2m} \tag{5}$$

$$V_\kappa = \sum_j \frac{\kappa}{2} x_j^2 \tag{6}$$

$$V_\gamma = \sum_j \frac{\gamma}{2}(x_j - x_{j-1})^2. \tag{7}$$

Calculate the contribution to $A, B, C, D$ from that one energies, and verify that there are no other terms necessary.

If you choose $K$ or $V_\kappa$, include terms which are up to linear order in $\alpha$, but neglect terms which are quadratic or higher. If you choose $V_\gamma$, neglect all terms which are linear or higher order in $\alpha$.

**Solution 1.1** (5 points)**.** As explained in the question, you need only do one of the following.

**Solution 1.1** (K).

$$K = \sum_j \frac{p_j^2}{2m} \tag{8}$$

$$= -\frac{\hbar^2}{4d^2m}\sum_j \left[\left(a_j - a_j^\dagger\right) - \alpha\left(a_{j+1} - a_{j+1}^\dagger + a_{j-1} - a_{j-1}^\dagger\right)\right]^2 \tag{9}$$

$$= -\frac{\hbar^2}{4d^2m}\sum_j \Bigg[\left(a_j - a_j^\dagger\right)^2 - \alpha\left(a_j - a_j^\dagger\right)\left(a_{j+1} - a_{j+1}^\dagger + a_{j-1} - a_{j-1}^\dagger\right) \tag{10}$$

$$- \alpha\left(a_{j+1} - a_{j+1}^\dagger + a_{j-1} - a_{j-1}^\dagger\right)\left(a_j - a_j^\dagger\right) + \mathcal{O}\left(\alpha^2\right)\Bigg]$$

$$= -\frac{\hbar^2}{4d^2m}\sum_j \left[\left(a_j a_j + a_j^\dagger a_j^\dagger\right) - \left(a_j a_j^\dagger + a_j^\dagger a_j\right) - 2\alpha\left(a_j - a_j^\dagger\right)\left(a_{j+1} - a_{j+1}^\dagger + a_{j-1} - a_{j-1}^\dagger\right)\right]. \tag{11}$$

The term proportional to $\alpha$ can be written:

$$\alpha \text{ term } = \frac{\hbar^2}{8d^2m}\sum_j 2\alpha\left(a_j - a_j^\dagger\right)\left(a_{j+1} - a_{j+1}^\dagger + a_{j-1} - a_{j-1}^\dagger\right) \tag{12}$$

$$= \frac{\alpha\hbar^2}{2d^2m}\left[\sum_j \left(a_j - a_j^\dagger\right)\left(a_{j+1} - a_{j+1}^\dagger\right) + \sum_j \left(a_j - a_j^\dagger\right)\left(a_{j-1} - a_{j-1}^\dagger\right)\right] \tag{13}$$

$$= \frac{\alpha\hbar^2}{2d^2m}\left[\sum_j \left(a_j - a_j^\dagger\right)\left(a_{j+1} - a_{j+1}^\dagger\right) + \sum_j \left(a_{j+1} - a_{j+1}^\dagger\right)\left(a_j - a_j^\dagger\right)\right] \tag{14}$$

$$= \frac{\alpha\hbar^2}{d^2m}\sum_j \left(a_j - a_j^\dagger\right)\left(a_{j+1} - a_{j+1}^\dagger\right) \tag{15}$$

$$= \sum_j \frac{\alpha\hbar^2}{d^2m}\left[\left(a_j a_{j+1} + a_j^\dagger a_{j+1}^\dagger\right) - \left(a_j^\dagger a_{j+1} + a_j a_{j+1}^\dagger\right)\right] \tag{16}$$

We have made liberal use of the relation $[a_{j+1}, a_j] = 0$. Therefore, we have:

$$A_K = \frac{\hbar^2}{4md^2}, \quad B_K = -\frac{\hbar^2\alpha}{d^2m}, \quad C_K = -\frac{\hbar^2}{4d^2m}, \quad D_K = \frac{\alpha\hbar^2}{d^2m}. \tag{17}$$

**Solution 1.1** ($V_\kappa$).

$$V_\kappa = \sum_j \frac{\kappa x_j^2}{2} \tag{18}$$

$$= \frac{\kappa d^2}{4} \sum_j \left[ \left(a_j + a_j^\dagger\right) - \alpha \left(a_{j+1} + a_{j+1}^\dagger + a_{j-1} + a_{j-1}^\dagger\right)\right]^2 \tag{19}$$

$$= \frac{\kappa d^2}{4} \sum_j \left[ \left(a_j + a_j^\dagger\right)^2 + 2\alpha \left(a_j + a_j^\dagger\right) \left(a_{j+1} + a_{j+1}^\dagger + a_{j-1} + a_{j-1}^\dagger\right) + \mathcal{O}\left(\alpha^2\right)\right] \tag{20}$$

$$= \frac{\kappa d^2}{4} \sum_j \left[ \left(a_j a_j + a_j^\dagger a_j^\dagger\right) + \left(a_j a_j^\dagger + a_j^\dagger a_j\right) + 4\alpha \left(a_j a_{j+1} + a_j^\dagger a_{j+1}^\dagger\right) + 4\alpha \left(a_j a_{j+1}^\dagger + a_j^\dagger a_{j+1}\right)\right]. \tag{21}$$

Therefore, we have:

$$K_{V_\kappa} = \frac{\kappa d^2}{4}, \quad B_{V_\kappa} = \kappa d^2 \alpha, \quad C_{V_\kappa} = \frac{\kappa d^2}{4}, \quad D_{V_\kappa} = \kappa d^2 \alpha. \tag{22}$$

**Solution 1.1** ($V_\gamma$).

$$V_\gamma = \sum_j \frac{\gamma}{2} \left(x_{j+1} - x_j\right)^2 \tag{23}$$

$$= \frac{\kappa d^2}{4} \sum_j \left(a_{j+1} + a_{j+1}^\dagger - a_j - a_j^\dagger + \mathcal{O}\left(\alpha^2\right)\right)^2 \tag{24}$$

$$= \frac{\kappa d^2}{2} \sum_j \left[\left(a_j^\dagger a_j + a_j a_j^\dagger\right) - \left(a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}\right) + \left(a_j a_j + a_j^\dagger a_j^\dagger\right) - \left(a_j a_{j+1} + a_{j+1}^\dagger a_j^\dagger\right)\right] \tag{25}$$

Therefore, we have:

$$K_{V_\gamma} = \frac{\gamma d^2}{2}, \quad B_{V_\gamma} = -\frac{\gamma d^2}{2}, \quad C_{V_\gamma} = \frac{\gamma d^2}{2}, \quad D_{V_\gamma} = -\frac{\gamma d^2}{2}. \tag{26}$$

**Problem 2. Classical Sound Waves**

Consider the model we studied in class:

$$H = \sum_j \frac{p_j^2}{2m} + \frac{\kappa}{2} x_j^2 + \frac{\gamma}{2}(x_j - x_{j-1})^2. \tag{27}$$

Lets take an infinite chain where $n$ extends from $-\infty$ to $+\infty$. The classical equations of motion for this model are

$$m\partial_t x_j = p_j \tag{28}$$

$$\partial_t p_j = -\kappa x_j - \gamma(2x_j - x_{j+1} - x_{j-1}). \tag{29}$$

In this problem you will solve these classical equations of motion, and find the frequencies $\omega$ of the classical modes. By the correspondence principle you expect the energy of of the quanta to be given by $\hbar\omega$.

To solve this classical problem, we guess a solution of the form $x_j = ue^{-i\omega t+ikaj}$, and $p_j = ve^{-i\omega t+ikaj}$ where $u, v, \omega$, and $j$ are constants, and the lattice spacing is $a$.

**2.1.** Substitute this ansatz into equation (28) and (29). You will have two equations with three unknowns $u, v, \omega$. Solve one of the equations for $u$, and substitute it into the other. You should get an equation which uniquely defines $\omega$.

[Alternatively, you can recognize your two equations as an eigenvalue problem for $\omega$. The above procedure is just a trick for solving that eigenvalue problem.]

**Solution 2.1** (3 points)**.** The two equations are:

$$-im\omega u = v \tag{30}$$

$$-i\omega v = -u\left(\kappa + \gamma\left(2 - e^{ika} - e^{-ika}\right)\right) = -u\left(\kappa + 2\gamma\left(1 - \cos\left(ka\right)\right)\right). \tag{31}$$

Plugging $u$ from the first into the second gives:

$$-i\omega v = \frac{v}{m\omega i}\left(\kappa + 2\gamma\left(1 - \cos\left(ka\right)\right)\right) \tag{32}$$

$$\omega^2 = \frac{1}{m}\left(\kappa + 2\gamma\left(1 - \cos\left(ka\right)\right)\right) \tag{33}$$

**2.2.** Taylor expand your expression for small $k$, to get an expression of the form $\omega^2 = A + Bk^2 + \cdots$. Given that classical frequencies are related to quantum mechanical energies, this is reminiscent of the relativistic formula $E^2 = p^2c^2 + M^4c^4$. By equating $E = \hbar\omega$, and $p = \hbar k$, find the "Mass gap" $M$ and the "effective speed of light $c$. Because of this analogy, these optical modes are sometimes called "Massive". [For contrast, the acoustic modes have no gap, and are called "massless."]

**Solution 2.2** (3 points)**.** For small k:

$$\omega^2 = \frac{1}{m}\left(\kappa + \gamma k^2 a^2\right) \tag{34}$$

$$E^2 = \hbar^2\omega^2 = \frac{\hbar^2\kappa}{m} + \frac{\gamma a^2}{m}\hbar^2 k^2. \tag{35}$$

Now, we know that $E^2 = p^2c^2 + M^2c^4 = \hbar^2k^2c^2 + M^2c^4$, and so from the second term we see that:

$$c^2 = \frac{\gamma a^2}{m} \tag{36}$$

and the first term gives us:

$$M^2 = \frac{\hbar^2 m\kappa}{\gamma^2 a^4}. \tag{37}$$

This mass gap means that there is a minimum energy we need to put in to excite any optical modes, as opposed to say, photons which can be produced with arbitrarily small energy (and arbitrarily large wavelength).

**Problem 3. Baryons**

Baryons are bound states of three quarks. They labeled by their quark composition, and their total angular momentum $J$. The"light" baryons are made of $u, d$, or $s$.

**3.1.** Look up the names of the 10 light baryons which have $J = 3/2$. List each of their quark composition.

**Solution 3.1** (3 points)**.**

| Quarks | Symbol | Quarks | Symbol |
|---|---|---|---|
| uuu | $\Delta^{++}$ | uds | $\Sigma^{*0}$ |
| uud | $\Delta^{+}$ | dds | $\Sigma^{*-}$ |
| udd | $\Delta^{0}$ | uss | $\Xi^{*0}$ |
| ddd | $\Delta^{-}$ | dss | $\Xi^{*-}$ |
| uus | $\Sigma^{*+}$ | sss | $\Omega^{-}$ |

**3.2.** Look up the names of the 8 light baryons which have $J = 1/2$. List each of their quark composition. (Note, both the neutral lambda and the neutral sigma have the same quark composition)

**Solution 3.2** (3 points)**.**

| Quarks | Symbol | Quarks | Symbol |
|---|---|---|---|
| uud | $p$ | uds | $\Sigma^{0}$ |
| udd | $n$ | dds | $\Sigma^{-}$ |
| uds | $\Lambda^{0}$ | uss | $\Xi^{0}$ |
| uus | $\Sigma^{+}$ | dss | $\Xi^{-}$ |

**3.3.** In a sentence or two, argue for why are there more light baryons with $J = 3/2$ than with $J = 1/2$? [Hint: think about the case $uuu$, and what the spin wavefunction looks like.]

**Solution 3.3** (2 points)**.** As explained in the lecture notes at `https://quantum.lassp.cornell.edu/lecture/elementary_particle_physics`, the baryon wavefunction has three parts: space, spin, flavor, and color. In order to form a bound state, the spatial part likely needs to be symmetric (think binding vs antibinding orbitals). The color part is always antisymetric. [Color is the "charge" of the strong force – and it is so strong that any particle seen in nature must be 11color neutral." For three particles, the only way to do this is to make it antisymmetric.] That leaves spin and flavor. Since quarks are fermions, the total wavefunction must be antisymmetric: thus either both spin and flavor are symmetric, or both spin and flavor are antisymmetric.
The $J = 3/2$ states are spin-symmetric (for example consider ↑↑↑,) and therefore they will be flavor symmetric. In particular you can have a $J = 3/2$, $uuu$ state. One is not, however, allowed to have a $uuu$ state which is $J = 1/2$ – that will not be consistent with fermionic symmetry.
Any argument that involves spin symmetry, and flavor symmetry will be accepted.

**Problem 4. Flavor Neutral Mesons**
In this question we will consider the flavor neutral mesons.

Mesons are bound states of a quark and anti-quark. A meson is "flavor neutral" if it has no net flavor – for example the bound state of and up and anti-up quark, or the bound state of a down and anti-down. Since these mesons all have the same "flavor" the strong interactions can mix them. That is, the energy eigenstate is a superposition of the various combinations.

The simplest model of these mesons is in terms of a three component wavefunction

$$|\psi\rangle = \psi_u|u\bar{u}\rangle + \psi_d|d\bar{d}\rangle + \psi_s|s\bar{s}\rangle, \tag{38}$$

where $|\psi_u|^2, |\psi_d|^2, |\psi_s|^2$ is the probability that the meson is in the $u\bar{u}, d\bar{d}$, or $s\bar{s}$ state.

The mass (energy) of the $\pi^0$, $\eta$ and $\eta'$ will be the eigenvalues of a 3-component Schrodinger equation

$$H\begin{pmatrix}\psi_u\\ \psi_d\\ \psi_s\end{pmatrix} = E\begin{pmatrix}\psi_u\\ \psi_d\\ \psi_s\end{pmatrix}, \tag{39}$$

where $H$ is a 3×3 matrix, and $E$ is identified with $M$. The eigenvectors give the flavor decomposition of these three mesons. Experimentally these mesons have mass 135MeV, 548MeV, and 958MeV.

**4.1.** A first stab at modeling $H$ is to assume that the standard model treats $u, d, s$ quarks all the same. We know that this is not completely true, but one can hypothesize that it is an approximate symmetry. One calls this (hypothetical) symmetry $SU(3)$ flavor symmetry – this sounds fancy, but it just means that the theory should be invariant under permuting the flavor among these 3 possibilities. The most general $H$ for this meson problem which treats $u, d$, and $s$ the same is

$$H_{SU(3)} = \begin{pmatrix} a & b & b\\ b & a & b\\ b & b & a\end{pmatrix}. \tag{40}$$

**Find the eigenvalues of this matrix.**

You should find that two of them are the same. This tells you that if $SU(3)$ flavor symmetry was exact, the mass of the $\pi^0$ should be equal to the mass of the $\eta$.

**Solution 4.1** (3 points)**.** This can be done in a number of ways. I like to simply guess the eigenvectors. Looking at the matrix, any of the following are eigenvectors: $(1,1,1)$ and $(1,0,-1)$, $(1,-1,0)$, and $(0,1,-1)$, with eigenvalues $a+2b$, $a-b$, $a-b$, and $a-b$. The latter 3 are not linearly independent – and the degenerate space is spanned by two eigenvectors. In particular, the standard basis for these are the $\pi^0$: $(1,-1,0)$ and the $\eta$: (1,1,-2). Regardless, the eigenvalues are

$$\text{Eigenvalues} = a+2b, \quad a-b, \quad a-b \tag{41}$$

The heavy one can be identified with the $\eta'$, while the two degenerate ones are the pion and $\eta$.

**4.2.** The fact that the $\pi_0$ and $\eta$ have different energies tells you that $SU(3)$ flavor symmetry is broken. One hypothesis is that you can account for the symmetry breaking solely by giving the $s$ quarks a higher mass. This would be modeled by a Hamiltonian

$$H_{qqs} = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & c \end{pmatrix}. \tag{42}$$

One could then fit $a, b, c$ to try to match the energies of the $\pi_0, \eta$ and $\eta'$ as closely as possible. I tried this procedure and came up with $a = 340\text{MeV}$, $b = 200\text{MeV}$, and $c = 938\text{MeV}$. Using these values for $a, b, c$, find the mass of the light flavor neutral mesons. [Feel free to use a computer algebra system.]

You should find this result is a lot better than the $SU(3)$ symmetric theory, but is still not perfect. Apparently the symmetry breaking can not all be accounted for by the mass of the strange quark. Given that we only have 3 pieces of data here (the masses), adding more parameters to our model is probably not wise.

**Solution 4.2** (3 points)**.** The eigenvectors are no longer obvious, but thankfully we can use a computer algebra system to find the eigenvalues.

$$\text{Eigenvalues} = 140\,\text{MeV}, \quad 393\,\text{MeV}, \quad 1085\,\text{MeV} \tag{43}$$

**4.3.** Within the model in Eq. (42) with the parameters $a = 340\text{MeV}$, $b = 200\text{MeV}$, and $c = 938\text{MeV}$, what are the flavor wavefunctions for the $\pi_0, \eta, \eta'$. [Again, you may want to use a conputer.] In what way do these differ from the nominal flavor wavefunctions listed in the meson tables?

**Solution 4.3** (3 points)**.** In the same order as the eigenvalues, we have the following normalised eigenvectors:

$$\begin{aligned}\text{Eigenvectors} &= (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), \quad (0.63, 0.63, -0.46), \quad (0.33, 0.33, 0.89)\\ &= \frac{1}{\sqrt{2}}|u\bar{u}\rangle - \frac{1}{\sqrt{2}}|d\bar{d}\rangle, \quad 0.63|u\bar{u}\rangle + 0.63|d\bar{d}\rangle - 0.46|s\bar{s}\rangle, \quad 0.33|u\bar{u}\rangle + 0.33|d\bar{d}\rangle + 0.89|s\bar{s}\rangle\end{aligned} \tag{44}$$

Meanwhile, the nominal mesons are

$$\begin{aligned}\pi^0 &= \frac{1}{\sqrt{2}}|u\bar{u}\rangle - \frac{1}{\sqrt{2}}|d\bar{d}\rangle\\ \eta &= \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle) = 0.41|u\bar{u}\rangle + 0.41|d\bar{d}\rangle - 0.82|s\bar{s}\rangle\\ \eta' &= \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle) = 0.58|u\bar{u}\rangle + 0.58|d\bar{d}\rangle + 0.58|s\bar{s}\rangle\end{aligned} \tag{45}$$

We see that $\pi^0$ matches very well, but not $\eta$ nor $\eta'$.

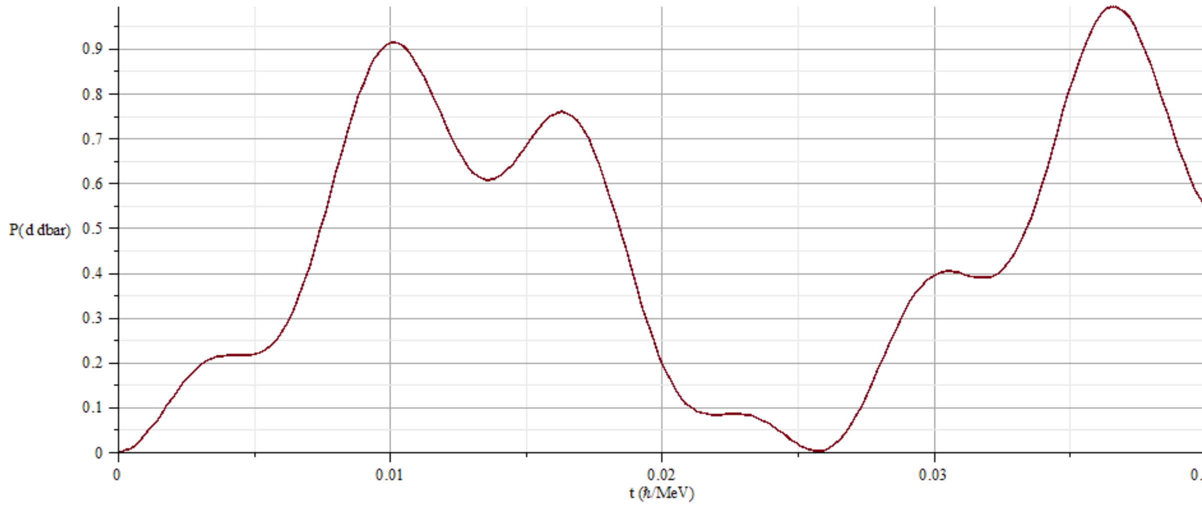
**4.4.** Suppose you created a meson in the $u\bar{u}$ state. Estimate what the characteristic time (in seconds) would be for it to oscillate into $d\bar{d}$? If the meson is moving at the speed of light, how far would it move (in meters) during that time? [You may find it useful to look back on the homework problem about the K mesons. You don't need to go through the whole argument here, just estimate the timescale.]

**Solution 4.4** (3 points)**.** Hopefully you have learnt from Homework 3 that the characteristic time for oscillation is $\sim \frac{\hbar\pi}{\Delta E}$ where $\Delta E$ is the difference between energy eigenvalues. Here, we have three energy eigenvalues, but since the two lightest eigenstates $\pi^0$ and $\eta$ contain more $|u\bar{u}\rangle$ and $|d\bar{d}\rangle$ than $\eta'$, the difference between the $\pi^0$ mass and $\eta$ mass is more relevant. Therefore, we crudely estimate that

$$\text{Timescale of oscillation} = \frac{\hbar\pi}{m_\eta - m_{\pi^0}} = \frac{\hbar\pi}{393\,\text{MeV} - 140\text{MeV}} = 8 \times 10^{-24}\,\text{s} \tag{46}$$

$$\text{Distance travelled (ignoring time dilation)} = 2 \times 10^{-15}\,\text{m} \tag{47}$$

Anything on this scale is fine (it did say to estimate).
Alternatively, one can exactly solve the problem. Here is a graph of the probability of begin in the $d\bar{d}$ state as a function of time.




Notice that the first major peak occurs at approximately $\hbar/\text{MeV} \sim 7 \times 10^{-24}\,\text{s}$ which is indeed very close to our estimate, although it is not a 100% oscillation. The shoulder-like feature at $0.004\,\hbar/\text{MeV}$ is related to the two other time scales $\frac{\hbar\pi}{m'_\eta - m_{\pi^0}}$ and $\frac{\hbar\pi}{m'_\eta - m_\eta}$ in the problem.

**Problem 5. Neutrino background**

In the early universe, the matter and photons were in thermal equilibrium. As the temperature fell to below $T \sim 10^{11}$K the neutrinos decoupled from the other particles. In this regime the neutrinos can be treated as noninteracting, massless, relativistic particles. Thus their energy density should be roughly the same as black-body radiation. [It will be a bit different because they are Fermions, but using the results for photons will not be too far off.]

**5.1.** Estimate the energy density in neutrinos at the time of decoupling. [It might help to look at your homework on the Planck Spectrum. Feel free to use results you derived there – you don't need to rederive the Plank Spectrum.] Quote your answer in $\text{eV/m}^3$. You can use the fact that

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}. \tag{48}$$

Include the fact that there are 3 flavors of neutrino, and 3 of antineutrino (but only one spin state of each).

**Solution 5.1** (5 points)**.** In the previous homework, for electromagnetic waves, we found the density of state in a box of sides $L$ to be (ignoring polarization)

$$\rho(\omega) = \frac{L^3\omega^2}{2\pi^2 c^3} \tag{49}$$

and the average energy of an electromagnetic mode to be

$$\langle E_\omega \rangle = \frac{\hbar\omega}{e^{\hbar\omega\beta} - 1} \tag{50}$$

The energy density is then given by

$$\begin{aligned} u_\gamma &= \frac{1}{L^3}\int_0^\infty d\omega \rho(\omega)\langle E_\omega\rangle \\ &= \frac{\hbar}{2\pi^2 c^3}\int_0^\infty d\omega \frac{\omega^3}{e^{\hbar\omega\beta}-1} \\ &= \frac{k_B^4}{2\pi^2\hbar^3 c^3}T^4\int_0^\infty dx \frac{x^3}{e^x - 1} \\ &= \frac{\pi^4}{15}\frac{k_B^4}{2\pi^2\hbar^3c^3}T^4. \end{aligned} \tag{51}$$

We will use the same expression for neutrinos (with the caveat that they are fermions, so this is an approximation). Including a flactof of 6 for the different flavors, we have

$$u_\nu = 6\frac{\pi^4}{15}\frac{k_B^4}{2\pi^2\hbar^3c^3}T^4 \tag{52}$$

$$= 1.97\frac{(8.62\cdot 10^{-5}eV/K)^4}{(6.5\cdot 10^{-16}eV\cdot s)^3(3\cdot 10^8 m.s)^3}(10^{11}K)^4 \tag{53}$$

$$= 1.5\times 10^{48}eV/m^3. \tag{54}$$

If we included the fact that the the neutrinos were fermions, we would have made the substitution

$$\frac{x^3}{e^x - 1} \longrightarrow \frac{x^3}{e^x + 1}. \tag{55}$$

It turns out that this does not make much of a difference because

$$\int_0^\infty \frac{x^3}{e^x+1}dx = \frac{7}{8}\frac{\pi^4}{15}, \tag{56}$$

and the answer would only change by a factor of 7/8.

**5.2.** The mass of a proton is roughly 1GeV. What density of Hydrogen atoms (in atoms per cubic meter) would you need so that the energy density from their rest mass is equal your result in 5.1. This will give you a feel for how big your answer is.

**Solution 5.2** (2 points)**.** We need a number density of

$$n_p = \frac{u}{m_p} = \frac{10^{48}\,\frac{\text{eV}}{\text{m}^3}}{10^9\,\text{eV}} = 10^{39}\,\text{m}^{-3}. \tag{57}$$

This is pretty darn big. It corresponds to a spacing between protons of $10^{-13}$m. This is lower than the density of matter inside of an atomic nucleus, but certainly higher than any conventional form of matter.

**5.3.** We actually know neutrinos have mass. Suppose the neutrino mass is about $m_\nu \sim 10^{-3}$eV. Assuming you have a gas of neutrinos at temperature $T$. At what temperature (in Kelvin) would you need to take into account the mass?

**Solution 5.3** (2 points)**.** Simply by dimensional analysis, we have to take the mass into account when $m_\nu c^2 \sim k_B T$

$$T \sim \frac{m_\nu c^2}{k_B} \sim 10\,\text{K} \tag{58}$$

which is 10 orders of magnitude smaller than freeze-out.

A more involved version of the argument is to compare the energy density from the mass of the neutrinos to the energy density we previously calculated. In particular, the density of neutrinos is

$$n_\nu = 6\frac{1}{L^3}\int_0^\infty d\omega \rho(\omega), \tag{59}$$

where $\rho(\omega)$ is the density of states defined in our previous homework. I didn't bother doing the integrals, but by dimensional analysis I know that if I equate $n_\nu m_\nu c^2$ and $u_\nu$, I will get $k_B T = m_\nu c^2$.

**Problem 6. Feedback**

**6.1.** How long did this homework take?

**6.2.** Which of the following words come to mind when you think about this homework (feel free to add your own words if you have something better): frustrating, fun, tedious, insightful, hard, easy, useful, useless, fair, unfair