P3317 HW from Lecture 20+21 and Recitation 12

Due Nov 20, 2018

Problem 1. Landau Levels

Here we will find the energy eigenstates a quantum mechanical particle moving in two dimensions in a uniform magnetic field. That is we will consider the Schrodinger equation

$$E\psi = -\frac{\hbar^2}{2m}\left(\nabla - iq\mathbf{A}\right)^2\psi, \tag{1}$$

where $\nabla \times \mathbf{A} = \mathbf{B}$. We will take $\mathbf{B}$ to point in the $\hat{\mathbf{z}}$ direction, and be uniform in space. The vector potential is not unique, we can add a divergence of any function to it. This is often described as "gauge freedom."

1.1. Show that

$$\mathbf{A} = \Lambda x\hat{\mathbf{y}},$$

where Λ is a constant, corresponds to a uniform magnetic field. This is known as the *Landau gauge*, and it is particularly useful for this calculation.

Solution 1.1 (2 points)**.**

$$\vec{B} = \nabla \times \vec{A} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} 0 \\ \Lambda x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \Lambda \end{pmatrix} = \Lambda\hat{z} \tag{2}$$

1.2. Within the Landau gauge, the Hamiltonian involves ∂_x, ∂_y, and x. It does not depend explicitly on y, so the momentum in the $\hat{y}$ direction is a good quantum number. Thus we write

$$\psi(x,y) = e^{ik_y y}\phi_{k_y}(x). \tag{3}$$

Plug this into Eq. (1) and get an equation for $\phi_{k_y}(x)$.

Solution 1.2 (3 points)**.**

$$\begin{aligned}
-\frac{1}{2m}(\nabla - iq\vec{A})^2\psi &= -\frac{1}{2m}\left[\nabla^2\psi - iq\nabla\cdot(\vec{A}\psi) - iq\vec{A}\cdot\nabla\psi - q^2\vec{A}\cdot\vec{A}\psi\right] \\
&= -\frac{1}{2m}\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - iq\frac{\partial(\Lambda x\psi)}{\partial y} - iq\Lambda x\frac{\partial\psi}{\partial y} - q^2\Lambda^2x^2\psi\right] \\
&= -\frac{1}{2m}\left[e^{ik_y y}\frac{\partial^2\phi_{k_y}(x)}{\partial x^2} + (-k_y^2 + 2k_y q\Lambda x - q^2\Lambda^2x^2)e^{ik_y y}\phi_{k_y}(x)\right] \\
&= Ee^{ik_y y}\phi_{k_y}(x)
\end{aligned} \tag{4}$$

So we have

$$-\frac{1}{2m}\left[\frac{\partial^2}{\partial x^2} - (k_y - q\Lambda x)^2\right]\phi_{k_y}(x) = E\phi_{k_y}(x) \tag{5}$$

1.3. You should recognize that this equation is the same equation obeyed by the wave function for a 1D quantum harmonic oscillator. What is the frequency ω_0 of the oscillator? Thinking about the classical motion of a charged particle in a magnetic field, what is the physical significance of this frequency?

Solution 1.3 (3 points)**.** Redefine $x' = x - \frac{k_y}{q\Lambda}$. Choosing the lazy and potentially confusing notation of $\phi_{k_y}(x') := \phi_{k_y}(x(x'))$, we get the 1D quantum harmonic oscillator equation:

$$-\frac{1}{2m}\frac{\partial^2 \phi_{k_y}(x')}{\partial x'^2} + \frac{1}{2}m\left(\frac{q\Lambda}{m}\right)^2 x'^2 \phi_{k_y}(x') = E\phi_{k_y}(x') \tag{6}$$

We see that $\omega_0 = \frac{q\Lambda}{m}$. Since $B_z = \Lambda$, this is the cyclotron frequency, the angular frequency of the circular orbit of a charged particle in a uniform magnetic field.

1.4. From your knowledge of the harmonic oscillator eigen-energies, what are the energies of all the single particle states? Note: since E does not depend on k_y, you have discovered a massive degeneracy.

Solution 1.4 (2 points)**.** The energies are $E_{(n,k_y)} = \hbar\omega_0\left(\frac{1}{2} + n\right)$, independent of k_y.

Problem 2. Guiding Centers

Consider a charged particle moving in 2D in a uniform magnetic field within the Landau Gauge:

$$H = \frac{1}{2m}\left[p_x^2 + (p_y - qBx)^2\right]. \tag{7}$$

Use Heisenberg equations of motion to calculate the rate of change of these four quantities: $X = \langle \hat{x} \rangle$, $Y = \langle \hat{y} \rangle$, $\Pi_x = \langle p_x \rangle$, and $\Pi_y = \langle \hat{p}_y - qB\hat{x} \rangle$. This should yield a set of four coupled differential equations which you can solve.

Hint 1: Start with the equations of motion for Π_x and Π_y. These will be closed. Solve them. Substitute the solutions into the equations for X and Y. Don't forget your constants of integration.
Hint 2: The Heisenberg equations of motion read:

$$\partial_t \langle \hat{O} \rangle(t) = \frac{1}{i\hbar}\int dr\, \psi^*(x,t)(\hat{O}\hat{H} - \hat{H}\hat{O})\psi(x,t) \tag{8}$$

$$= \frac{1}{i\hbar}\langle [\hat{O}, \hat{H}] \rangle. \tag{9}$$

Hint 3: In case 2D confuses you – recall x commutes with y and p_y.

Hint 4: Recall the following:

$$
\begin{aligned}
[\hat{x}, \hat{p}] &= i\hbar & (10) \\
[\hat{x}, \hat{p}^2] &= 2i\hbar\hat{p} & (11) \\
[\hat{x}, \hat{x}] &= 0 & (12) \\
[\hat{x}, \hat{x}^2] &= 0 & (13) \\
[\hat{p}, \hat{x}] &= -i & (14) \\
[\hat{p}, \hat{x}^2] &= -2i\hbar\hat{x} & (15) \\
[\hat{p}, \hat{p}] &= 0 & (16) \\
[\hat{p}, \hat{p}^2] &= 0. & (17)
\end{aligned}
$$

Solution 2.1 (8 points)**.** The commutators with the Hamiltonian are

$$\left[\hat{x}, \hat{H}\right] = \frac{1}{2m}\left[\hat{x}, \hat{p}_x^2\right] = i\hbar\frac{\hat{p}_x}{m} \tag{18}$$

$$\left[\hat{y}, \hat{H}\right] = \frac{1}{2m}\left[\hat{y}, \hat{p}_y^2\right] - \frac{2qB}{2m}\left[\hat{y}, \hat{p}_y\right]\hat{x} = i\hbar\frac{\hat{p}_y}{m} - i\hbar\frac{qB}{m}\hat{x} \tag{19}$$

$$\left[\hat{p}_x, \hat{H}\right] = -\frac{2qB}{2m}\left[\hat{p}_x, \hat{x}\right]\hat{p}_y + \frac{q^2B^2}{2m}\left[\hat{p}_x, \hat{x}^2\right] = i\hbar\frac{qB}{m}\hat{p}_y - i\hbar\frac{q^2B^2}{m}\hat{x} \tag{20}$$

$$\left[\hat{p}_y - qB\hat{x}, \hat{H}\right] = \left[\hat{p}_y, \hat{H}\right] - \left[qB\hat{x}, \hat{H}\right] = 0 - i\hbar\frac{qB}{m}\hat{p}_x \tag{21}$$

Taking expectation values on both sides, we then get the following equations of motion:

$$\begin{aligned} \frac{dX}{dt} &= \frac{\Pi_x}{m} \\ \frac{dY}{dt} &= \frac{\Pi_y}{m} \\ \frac{d\Pi_x}{dt} &= \frac{qB}{m}\Pi_y \\ \frac{d\Pi_y}{dt} &= -\frac{qB}{m}\Pi_x \end{aligned} \tag{22}$$

Let's solve these differential equations. Define $\omega = \frac{qB}{m}$. The last two equations can be combined to give

$$\begin{aligned} \frac{d^2\Pi_x}{dt^2} + \omega^2\Pi_x = 0 \\ \frac{d^2\Pi_y}{dt^2} + \omega^2\Pi_y = 0 \end{aligned} \tag{23}$$

The general solution is

$$\begin{aligned} \Pi_x &= a\cos(\omega t) + b\sin(\omega t) \\ \Pi_y &= b\cos(\omega t) - a\sin(\omega t) \end{aligned} \tag{24}$$

Note that the coefficients of Π_y are not arbitrary relative to those of Π_x, because Π_x and Π_y must still satisfy the last two equations in the original system of equations.

We now solve for X and Y, using the first two equations in the original system. We find that

$$\begin{aligned} X &= c + \frac{a}{m\omega}\sin(\omega t) - \frac{b}{m\omega}\cos(\omega t) \\ Y &= d + \frac{b}{m\omega}\sin(\omega t) + \frac{a}{m\omega}\cos(\omega t) \end{aligned} \tag{25}$$

This describes a circular motion, centered at (c, d) of radius $\frac{\sqrt{a^2+b^2}}{m\omega}$, with angular frequency ω.

Problem 3. Maxwell's Equations

We define the electromagnetic field tensor as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{26}$$

where $\mu, \nu = t, x, y, z$, and $A_t = -\phi$ is the scalar potential, and A_x, A_y, A_z are the components of the vector potential.

3.1. What are the following in terms of E and B: $F_{xt}, F_{yt}, F_{zt}, F_{xy}, F_{yz}, F_{zx}$?

Solution 3.1 (4 points)**.** The electric field involved space-time loops, while the magnetic field involves spatial loops:

$$\begin{pmatrix} F_{xt} \\ F_{yt} \\ F_{zt} \end{pmatrix} = \begin{pmatrix} \partial_x A_t - \partial_t A_x \\ \partial_y A_t - \partial_t A_y \\ \partial_z A_t - \partial_t A_z \end{pmatrix} = -\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \phi - \partial t \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = -\nabla\phi - \partial_t \vec{A} = \vec{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{27}$$

$$\begin{pmatrix} F_{yz} \\ F_{zx} \\ F_{xy} \end{pmatrix} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix} = \nabla \times \vec{A} = \vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \tag{28}$$

3.2. Consider the quantity

$$\Lambda_{xyz} = \partial_x F_{yz} + \partial_y F_{zx} + \partial_z F_{xy}. \tag{29}$$

By writing F in terms of A, show that $\Lambda_{xyz} = 0$. This should be part of one of Maxwell's equations. Which one is it?

Solution 3.2 (2 points)**.**

$$\begin{aligned} \Lambda_{xyz} &= \partial_x F_{yz} + \partial_y F_{zx} + \partial_z F_{xy} \\ &= \partial_x(\partial_y A_z - \partial_z A_y) + \partial_y(\partial_z A_x - \partial_x A_z) + \partial_z(\partial_x A_y - \partial_y A_x) \\ &= 0 \end{aligned} \tag{30}$$

This corresponds to $\nabla \cdot \vec{B} = 0$.

3.3. Consider the quantity

$$\Lambda_{txy} = \partial_t F_{xy} + \partial_x F_{yt} + \partial_y F_{tx}. \tag{31}$$

By writing F in terms of A, show that $\Lambda_{txy} = 0$. This should be part of one of Maxwell's equations. You can complete that equation by looking at Λ_{tyz} and Λ_{tyz} (but you do not need to).

Solution 3.3 (2 points)**.**

$$\begin{aligned} \Lambda_{txy} &= \partial_t F_{xy} + \partial_x F_{yt} + \partial_y F_{tx} \\ &= \partial_t(\partial_x A_y - \partial_y A_x) + \partial_x(\partial_y A_t - \partial_t A_y) + \partial_y(\partial_t A_x - \partial_x A_t) \\ &= 0 \end{aligned} \tag{32}$$

This corresponds to part of $\partial_t \vec{B} + \nabla \times \vec{E} = 0$.

We will not prove it here, but it turns out that in this notation the other two Maxwell equations are

$$\partial_x F_{xt} + \partial_y F_{yt} + \partial_z F_{zt} = 0 \tag{33}$$
$$-\partial_t F_{tx} + \partial_y F_{yx} + \partial_z F_{zx} = 0 \tag{34}$$
$$-\partial_t F_{ty} + \partial_x F_{xy} + \partial_z F_{zy} = 0 \tag{35}$$
$$-\partial_t F_{tz} + \partial_x F_{xz} + \partial_y F_{yz} = 0 \tag{36}$$
$$\tag{37}$$

Problem 4. Goldstone mode

Consider the following wave equation

$$\partial_t^2 \phi - \partial_x^2 \phi - m^2 \phi + \lambda^2 |\phi|^2 \phi = 0. \tag{38}$$

This is *not* a gauge theory.

4.1. Linearize Eq. (38) about $\phi = \phi_0 = m/\lambda$.

Solution 4.1 (4 points)**.**

$$\partial_t^2 (\phi_0 + a + ib) - \partial_x^2 (\phi_0 + a + ib) - m^2 (\phi_0 + a + ib) + \lambda^2 |\phi_0 + a + ib|^2 (\phi_0 + a + ib) = 0 \tag{39}$$
$$\partial_t^2 a + i\partial_t^2 b - \partial_x^2 a - i\partial_x^2 b - m^2 (a + ib) + 3m^2 a + im^2 b = 0. \tag{40}$$

Taking the real and imaginary parts gives:

$$\left(\partial_t^2 - \partial_x^2 + 2m^2\right) a = 0 \tag{41}$$
$$\left(\partial_t^2 - \partial_x^2\right) b = 0. \tag{42}$$

4.2. What are the spectra of excitations? You should find one massive (gapped) mode. You will also find a gapless mode – the Goldstone mode.

Solution 4.2 (4 points)**.** Substituting in $a = a_0 e^{ikx - i\omega t}$ gives a dispersion relation:

$$\omega^2 = k^2 + 2m^2 \tag{43}$$

implying that this mode has a mass $\sqrt{2}m$, while for b we get:

$$\omega^2 = k^2 \tag{44}$$

which is a massless mode.

This is a generic result. Whenever you have a continuously degenerate ground state, you will have a gapless mode. For example, in a crystal there is a continuous degeneracy corresponding to where the first atom sits. Once you specify its location you know where all of them are. Thus a crystal should have a gapless mode. It in fact has 3 gapless modes: the two transverse acoustic phonons, and a single longitudinal acoustic phonon.

Problem 5. Feedback

5.1. How long did this homework take?

5.2. Which of the following words come to mind when you think about this homework (feel free to add your own words if you have something better): frustrating, fun, tedious, insightful, hard, easy, useful, useless, fair, unfair